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POSITIVE SOLUTIONS OF A SINGULAR FRACTIONAL BOUNDARY VALUE PROBLEM WITH A FRACTIONAL BOUNDARY CONDITION

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Abstract. For $\alpha \in (1, 2]$, the singular fractional boundary value problem

$$D^{\alpha}_{0+}x + f\left(t, x, D^{\mu}_{0+}x\right) = 0,\ \ 0 < t < 1,$$

satisfying the boundary conditions $x(0) = D^{\beta}_{0+}x(1) = 0$, where $\beta \in (0, \alpha - 1]$, $\mu \in (0, \alpha - 1]$, and D^{α}_{0+}, D^{β}_{0+} and D^{μ}_{0+} are Riemann-Liouville derivatives of order α, β and μ respectively, is considered. Here f satisfies a local Carathéodory condition, and $f(t, x, y)$ may be singular at the value 0 in its space variable x. Using regularization and sequential techniques and Krasnosel'skii's fixed point theorem, it is shown this boundary value problem has a positive solution. An example is given.

Keywords: fractional differential equation, singular problem, fixed point.

Mathematics Subject Classification: 26A33, 34A08, 34B16.

1. INTRODUCTION

For $\alpha \in (1, 2]$, we consider the singular fractional boundary value problem

$$D^{\alpha}_{0+}x + f\left(t, x, D^{\mu}_{0+}x\right) = 0, \quad 0 < t < 1, \tag{1.1}$$

satisfying the boundary conditions

$$x(0) = D^{\beta}_{0+}x(1) = 0, \tag{1.2}$$

where $\beta \in (0, \alpha - 1]$, $\mu \in (0, \alpha - 1]$, and D^{α}_{0+}, D^{β}_{0+} and D^{μ}_{0+} are Riemann-Liouville derivatives of order α, β and μ respectively. Here f satisfies the local Carathéodory condition on $[0, 1] \times \mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, ($f \in \mathrm{Car}([0, 1] \times \mathcal{D})$)and $f(t, x, y)$ may be singular at the value 0 in its space variable x. By a positive solution, we mean x satisfies (1.1), (1.2) and $x(t) > 0$ for $t \in (0, 1]$.

The study of fractional boundary value problems has seen a tremendous expansion in recent years motivated by both general theory and physical representations and applications. For the reader interested in such works, we refer to [2, 4, 7, 8]. Of interest to the work presented, we point to research investigating the existence of solutions to fractional boundary value problems [1, 6, 9–12].

In [1], the authors proved the existence of at least one positive solution to the Dirichlet boundary value problem

$$D_{0+}^{\alpha} x + f(t, x, D_{0+}^{\mu} x) = 0,$$
$$x(0) = x(1) = 0$$

with $\alpha \in (1, 2)$, $\mu > 0$ and $\alpha - \mu \geq 1$ using Green's functions and the Krasnosel'skii fixed point theorem after placing certain conditions upon f.

Our aim in this work is to use the same differential equation, but instead of Dirichlet boundary conditions, we incorporate fractional boundary conditions, $x(0) = D_{0+}^{\beta} x(1) = 0$ with $\beta \in (0, \alpha - 1]$. Recently, the Green's function for (1.1), (1.2) was found in [3] which affords us the opportunity to utilize operators and an application of Krasnosel'skii's fixed point theorem . Since f might have a singularity in the function space at $x = 0$, we must also use regularization and sequential techniques.

In section 2, we introduce definitions, assumptions, and define a sequence of functions, $\{f_n\}$, to handle the possible singularity at $x = 0$. Section 3 is where one will find the Green's function and its associated properties along with the Krasnosel'skii fixed point theorem. Additionally, we prove the existence of a sequence of positive solutions, $\{x_n(t)\}$, to the auxiliary problem. Finally, in section 4, we make the jump from a sequence of auxiliary solutions to a positive solution $x(t)$ of (1.1), (1.2). We conclude with an example.

2. PRELIMINARY DEFINITIONS AND ASSUMPTIONS

We start with the definition of the Riemann-Liouville fractional integral and fractional derivative. Let $\nu > 0$. The Riemann-Liouville fractional integral of a function x of order ν, denoted $I_{0+}^{\nu} u$, is defined as

$$I_{0+}^{\nu} x(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} x(s) ds,$$

provided the right-hand side exists. Moreover, let n denote a positive integer and assume $n - 1 < \alpha \leq n$. The Riemann-Liouville fractional derivative of order α of the function $x : [0, 1] \to \mathbb{R}$, denoted $D_{0+}^{\alpha} x$, is defined as

$$D_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} x(s) ds = D^n I_{0+}^{n-\alpha} x(t),$$

provided the right-hand side exists.

We will make use of the power rule, which states that [2]

$$D_{0+}^{\nu_2} t^{\nu_1} = \frac{\Gamma(\nu_1 + 1)}{\Gamma(\nu_1 + 1 - \nu_2)} t^{\nu_1 - \nu_2}, \quad \nu_1 > -1, \nu_2 \geq 0, \tag{2.1}$$

where it is assumed that $\nu_2 - \nu_1$ is not a positive integer. If $\nu_2 - \nu_1$ is a positive integer, then the right hand side of (2.1) vanishes. To see this, one can appeal to the convention that $\frac{1}{\Gamma(\nu_1+1-\nu_2)} = 0$ if $\nu_2 - \nu_1$ is a positive integer, or one can perform the calculation on the left hand side and calculate

$$D^n t^{n-(\nu_2-\nu_1)} = 0.$$

We say that f satisfies the local Carathéodory condition on $[0,1] \times \mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, if

1. $f(\cdot, x, y) : [0,1] \to \mathbb{R}$ is measurable for all $(x, y) \in \mathcal{D}$;
2. $f(t, \cdot, \cdot) : \mathcal{D} \to \mathbb{R}$ is continuous for a.e. $t \in [0,1]$; and
3. for each compact set $\mathcal{H} \subset \mathcal{D}$, there is a function $\varphi_{\mathcal{H}} \in L^1[0,1]$ such that
$$|f(t, x, y)| \leq \varphi_{\mathcal{H}}(t),$$
for a.e. $t \in [0,1]$ and all $(x, y) \in \mathcal{H}$.

Throughout the paper,

$$\|x\|_L = \int_0^1 |x(t)| dt, \qquad \|x\|_0 = \max_{t \in [0,1]} |x(t)|,$$

and

$$\|x\| = \max\{\|x\|_0, \|D_{0+}^{\mu} x\|_0\}.$$

We assume the following conditions on f.

(H1) $f \in \operatorname{Car}([0,1] \times \mathcal{D})$, $\mathcal{D} = (0, \infty) \times \mathbb{R}$,

$$\lim_{x \to 0^+} f(t, x, y) = \infty,$$

for a.e. $t \in [0,1]$ and all $y \in \mathbb{R}$, and there exists a positive constant m such that, for a.e. $t \in [0,1]$ and all $(x, y) \in \mathcal{D}$,

$$f(t, x, y) \geq m.$$

(H2) f satisfies the estimate for a.e. $t \in [0,1]$ and all $(x, y) \in \mathcal{D}$,

$$f(t, x, y) \leq \gamma(t) \left(q(x) + p(x) + \omega(|y|) \right),$$

where $\gamma \in L^1[0,1]$, $q \in C(0, \infty)$, and $p, \omega \in C[0, \infty)$ are positive, q is nonincreasing, p and ω are nondecreasing, and

$$\int_0^1 \gamma(t) q(M t^{\alpha-1}) dt < \infty, \quad M = \frac{m\beta}{(\alpha - \beta)\Gamma(\alpha + 1)},$$

$$\lim_{x \to \infty} \frac{p(x) + \omega(x)}{x} = 0.$$

We use regularization and sequential techniques to show the existence of solutions of (1.1), (1.2). Thus, for $n \in \mathbb{N}$, define f_n by

$$f_n(t,x,y) = \begin{cases} f(t,x,y), & x \geq 1/n, \\ f\left(t, \frac{1}{n}, y\right) & x < 1/n, \end{cases}$$

for a.e. $t \in [0,1]$ and for all $(x,y) \in \mathcal{D}_* := [0,\infty) \times \mathbb{R}$. Then $f_n \in \mathrm{Car}([0,1] \times \mathcal{D}_*)$,

$$f_n(t,x,y) \geq m,$$

for a.e. $t \in [0,1]$ and all $(x,y) \in \mathcal{D}_*$,

$$f_n(t,x,y) \leq \gamma(t)(q(1/n) + p(x) + p(1) + \omega(|y|)),$$

for a.e. $t \in [0,1]$ and all $(x,y) \in \mathcal{D}_*$, and

$$f_n(t,x,y) \leq \gamma(t)(q(x) + p(x) + p(1) + \omega(|y|)),$$

for a.e. $t \in [0,1]$ and all $(x,y) \in \mathcal{D}$.

3. POSITIVE SOLUTIONS OF THE AUXILIARY PROBLEM

To use these techniques, we first discuss solutions of the fractional differential equation

$$D^{\alpha}_{0^+} x + f_n(t, x, D^{\mu}_{0^+} x) = 0, \quad 0 < t < 1, \tag{3.1}$$

satisfying boundary conditions (1.2).

The Green's function for $-D^{\alpha}_{0^+} u = 0$ satisfying the boundary conditions (1.2) is given by (see [3])

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t < 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}, & 0 \leq t \leq s < 1. \end{cases} \tag{3.2}$$

Therefore, x is a solution of (3.1), (1.2) if and only if

$$x(t) = \int_0^1 G(t,s) f_n(s, x(s), D^{\mu}_{0^+} x(s)) ds, \quad 0 \leq t \leq 1.$$

Lemma 3.1. *Let G be defined as in* (3.2). *Then*

1. $G(t,s) \in C([0,1] \times [0,1])$ *and* $G(t,s) > 0$ *for* $(t,s) \in (0,1) \times (0,1)$*;*
2. $G(t,s) \leq \frac{1}{\Gamma(\alpha)}$ *for* $(t,s) \in [0,1] \times [0,1]$*; and*
3. $\int_0^1 G(t,s) ds \geq \dfrac{\beta t^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha+1)}$ *for* $t \in [0,1]$.

Proof.

1. G is continuous by definition. The proof that $G(t,s) > 0$ for $(t,s) \in (0,1) \times (0,1)$ can be found in [3].
2. Next, we remark that since $0 \le t \le 1$ and $\alpha > 1$, $t^{\alpha-1} \le 1$. Also, notice that since $0 \le \beta \le \alpha - 1$ and $0 \le s \le 1$, $(1-s)^{\alpha-1-\beta} \le 1$. So $G(t,s) \le \frac{1}{\Gamma(\alpha)}$ for $(t,s) \in [0,1] \times [0,1]$.
3. Now, for $t \in [0,1]$,

$$\begin{aligned}\int_0^1 G(t,s)ds &= \int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} ds \\ &= \frac{1}{\Gamma(\alpha)} \left(t^{\alpha-1} \int_0^1 (1-s)^{\alpha-1-\beta} ds - \int_0^t (t-s)^{\alpha-1} \right) \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)} \frac{\alpha - t(\alpha-\beta)}{\alpha(\alpha-\beta)}.\end{aligned}$$

But for $t \in [0,1]$, $\alpha - (t\alpha - \beta) > \beta$. Therefore,

$$\begin{aligned}\int_0^1 G(t,s)ds &= \frac{t^{\alpha-1}}{\Gamma(\alpha)} \frac{\alpha - t(\alpha-\beta)}{\alpha(\alpha-\beta)} \\ &\ge \frac{\beta t^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha+1)},\end{aligned}$$

for $t \in [0,1]$. □

Define

$$Q_n x(t) = \int_0^1 G(t,s) f_n(s, x(s), D_{0^+}^{\mu} x(s)) ds, \quad 0 \le t \le 1.$$

Let $X = \{x \in C[0,1] : D_{0^+}^{\mu} x \in C[0,1]\}$ with norm $\|\cdot\|$ defined earlier. Notice X is a Banach space. Define a cone $\mathcal{P}$ in X as

$$\mathcal{P} = \{x \in X : x(t) \ge 0 \text{ for } t \in [0,1]\}.$$

Note if $x \in \mathcal{P}$ is a fixed point of Q_n, then x is a positive solution of (3.1), (1.2). To that end, we will use the well-known Krasnosel'skii Fixed Point Theorem, which is stated below, to show the existence of positive solutions of (3.1), (1.2).

Theorem 3.2 (Krasnosel'skii's Fixed Point Theorem [5])**.** *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset X$ be a cone in $\mathcal{P}$. Assume that Ω_1, Ω_2 are open sets with $0 \in \Omega_1$, and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$ be a completely continuous operator such that*

$$\|Tu\| \ge \|u\|, \ u \in \mathcal{P} \cap \partial\Omega_1, \ \text{and} \ \|Tu\| \le \|u\|, \ u \in \mathcal{P} \cap \partial\Omega_2.$$

Then T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Lemma 3.3. *Let* (H1) *and* (H2) *hold. Then* $Q_n : \mathcal{P} \to \mathcal{P}$ *and* Q_n *is a completely continuous operator.*

Proof. Suppose that $x \in \mathcal{P}$. Then,

$$Q_n x(t) = \int_0^1 G(t,s) f_n(s, x(s), D_{0+}^{\mu} x(s)) ds.$$

From Lemma 3.1 (1.), $G(t,s)$ is continuous and nonnegative on $[0,1] \times [0,1]$. So $Q_n x \in C[0,1]$. Also, by using (2.1),

$$\begin{aligned}(D_{0+}^{\mu} Q_n)x(t) = \frac{1}{\Gamma(\alpha-\mu)} \Bigg(& t^{\alpha-\mu-1} \int_0^1 (1-s)^{\alpha-\beta-1} f_n\left(s, x(s), D_{0+}^{\mu} x(s)\right) ds \\ & - \int_0^t (t-s)^{\alpha-\mu-1} f_n\left(s, x(s), D_{0+}^{\mu} x(s)\right) ds \Bigg),\end{aligned}$$

and so $D_{0+}^{\mu} Q_n x \in C[0,1]$. So $Q_n : X \to X$. By (H1) and the definition of $f_n(t,x,y)$, we have $f_n\left(s, x(s), D_{0+}^{\mu} x(s)\right) \geq m > 0$ for a.e. $t \in [0,1]$. Therefore, for $x \in \mathcal{P}$, Lemma 3.1 (1.) gives that $Q_n x(t) \geq 0$ for $t \in [0,1]$. Thus, $Q_n : \mathcal{P} \to \mathcal{P}$.

Next, we show that Q_n is a continuous operator. To that end, let $\{x_k\} \subset \mathcal{P}$ be a convergent sequence such that $\lim_{k\to\infty} \|x_k - x\| = 0$. Then, $\lim_{k\to\infty} x_k(t) = x(t)$ uniformly on $[0,1]$ and $\lim_{k\to\infty} D_{0+}^{\mu} x_k(t) = D_{0+}^{\mu} x(t)$ uniformly on $[0,1]$. Also, $x \in \mathcal{P}$.

Let

$$\rho_k(t) = f_n\left(t, x_k(t), D_{0+}^{\mu} x_k(t)\right), \quad \rho(t) = f_n(t, x(t), D_{0+}^{\mu} x(t)).$$

Then, $\lim_{k\to\infty} \rho_k(t) = \rho(t)$ for a.e. $t \in [0,1]$. Since $f_n \in \mathrm{Car}([0,1] \times \mathbb{R}^2)$ and $\{x_k\}$ and $\{D_{0+}^{\mu} x_k\}$ are bounded in $C[0,1]$, there exists $\varphi \in L^1[0,1]$ such that $m \leq \rho_k(t) \leq \varphi(t)$ for a.e. $t \in [0,1]$ and all $k \in \mathbb{N}$. By the Lebesgue Dominated Convergence Theorem,

$$\lim_{k\to\infty} \int_0^1 |\rho_k(s) - \rho(s)| ds = 0.$$

By Lemma 3.1 (2.),

$$|(Q_n x_k)(t) - (Q_n x)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^1 |\rho_k(s) - \rho(s)| ds.$$

Therefore, $\lim_{k\to\infty}(Q_n x_k)(t) = (Q_n x)(t)$ uniformly for $t \in [0,1]$. Also,

$$\begin{aligned}|(D^{\mu}_{0+}Q_n x_k)(t) - (D^{\mu}_{0+}Q_n x)(t)| &\le \frac{1}{\Gamma(\alpha-\mu)}\left(t^{\alpha-\mu-1}\int_0^1 (1-s)^{\alpha-\beta-1}|\rho_k(s)-\rho(s)|ds\right.\\ &\quad \left.+ \int_0^t (t-s)^{\alpha-\mu-1}|\rho_k(s)-\rho(s)|ds\right)\\ &\le \frac{2}{\Gamma(\alpha-\mu)}\int_0^1 |\rho_k(s)-\rho(s)|ds.\end{aligned}$$

So, $\lim_{k\to\infty}(D^{\mu}_{0+}Q_n x_k)(t) = (D^{\mu}_{0+}Q_n x)(t)$ uniformly for $t \in [0,1]$. Thus, $\|Q_n x_k - Q_n x\| \to 0$ and hence, Q_n is a continuous operator.

For $W \in \mathbb{R}^+$, define $\mathcal{W} = \{x \in \mathcal{P} : \|x\| \le W\}$ to be a bounded subset of $\mathcal{P}$. Let ρ be as before. Then there exists a $\varphi \in L^1[0,1]$ with $m \le \rho(t) \le \varphi(t)$ for a.e. $t \in [0,1]$ as before. Since, for $x \in \mathcal{W}$,

$$|(Q_n x)(t)| \le \frac{1}{\Gamma(\alpha)}\int_0^1 \varphi(s)ds = \frac{\|\varphi\|_1}{\Gamma(\alpha)},$$

and

$$|(D^{\mu}_{0+}Q_n x)(t)| \le \frac{2}{\Gamma(\alpha-\mu)}\int_0^1 \varphi(s)ds = \frac{2\|\varphi\|_1}{\Gamma(\alpha-\mu)},$$

it follows that $\{Q_n x : x \in \mathcal{W}\}$ and $\{D^{\mu}_{0+}Q_n x : x \in \mathcal{W}\}$ are uniformly bounded. Next, let $0 \le t_1 < t_2 \le 1$. Then for $x \in \mathcal{W}$,

$$\begin{aligned}|Q_n x(t_2) - Q_n x(t_1)| &\le \frac{1}{\Gamma(\alpha)}\left(\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)\int_0^1 (1-s)^{\alpha-1-\beta}\varphi(s)ds\right.\\ &\quad + \int_0^{t_1}\left((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right)\varphi(s)ds\\ &\quad \left.+ (t_2-t_1)^{\alpha-1}\int_{t_1}^{t_2}\varphi(s)ds\right)\end{aligned}$$

and

$$
\begin{aligned}
&|(D^{\mu}_{0^+}Q_n x)(t_2) - (D^{\mu}_{0^+}Q_n x)(t_1)| \\
&\leq \frac{1}{\Gamma(\alpha-\mu)}\left(\left(t_2^{\alpha-\mu-1} - t_1^{\alpha-\mu-1}\right)\int_0^1 (1-s)^{\alpha-\beta-1}\varphi(s)ds\right. \\
&\quad \left. + \int_0^{t_1}\left((t_2-s)^{\alpha-\mu-1} - (t_1-s)^{\alpha-\mu-1}\right)\varphi(s)ds + (t_2-t_1)^{\alpha-\mu-1}\int_{t_1}^{t_2}\varphi(s)ds\right).
\end{aligned}
$$

Thus, with the appropriate choice of δ, it can be shown that for $\epsilon > 0$, if $t_2 - t_1 < \delta$, $|Q_n x(t_2) - Q_n x(t_1)| < \epsilon$ and $|(D^{\mu}_{0^+}Q_n x)(t_2) - (D^{\mu}_{0^+}Q_n x)(t_1)| < \epsilon$. Therefore, $\{Q_n x : x \in \mathcal{W}\}$ and $\{D^{\mu}_{0^+}Q_n x : x \in \mathcal{W}\}$ are equicontinuous, and by the Arzelà-Ascoli theorem, Q_n is a completely continuous operator. □

Lemma 3.4. *Let* (H1) *and* (H2) *hold. Then* (3.1)*,* (1.2) *has a positive solution* x^* *with* $x^*(t) \geq Mt^{\alpha-1}$ *for* $t \in [0,1]$*.*

Proof. Define $\Omega_1 = \{x \in X : \|x\| < M\}$. Then for $x \in P \cap \partial\Omega_1$ and $t \in [0,1]$,

$$
(Q_n x)(t) = \int_0^1 G(t,s) f_n\left(s, x(s), D^{\mu}_{0^+}x(s)\right) \geq m\int_0^1 G(t,s) \geq Mt^{\alpha-1}.
$$

So $\|Q_n x\|_0 \geq M$. Consequently, $\|Q_n x\| \geq \|x\|$ for $x \in P \cap \partial\Omega_1$.

Next, notice that for $x \in \mathcal{P}$ and $t \in [0,1]$,

$$
\begin{aligned}
|(Q_n x)(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^1 \gamma(s)\left(q(1/n) + p(x(s)) + p(1) + \omega\left(\left|D^{\mu}_{0^+}x(s)\right|\right)\right) \\
&\leq \frac{1}{\Gamma(\alpha)}\left(q(1/n) + p(\|x\|_0) + p(1) + \omega\left(\left\|D^{\mu}_{0^+}x\right\|_0\right)\right)\|\gamma\|_L.
\end{aligned}
$$

Also, for $x \in \mathcal{P}$,

$$
\begin{aligned}
|D^{\mu}_{0^+}(Q_n x)(t)| &= \left|\frac{1}{\Gamma(\alpha-\mu)}\left(t^{\alpha-\mu-1}\int_0^1 (1-s)^{\alpha-\beta-1} f_n\left(s, x(s), D^{\mu}_{0^+}x(s)\right)\right.\right. \\
&\quad \left.\left. - \int_0^t (t-s)^{\alpha-\mu-1} f_n\left(s, x(s), D^{\mu}_{0^+}x(s)\right)\right)\right| \\
&\leq \frac{2}{\Gamma(\alpha-\mu)}\left(q(1/n) + p(\|x\|_0) + p(1) + \omega\left(\left\|D^{\mu}_{0^+}x\right\|_0\right)\right)\|\gamma\|_L.
\end{aligned}
$$

So for $K = \max\left\{\frac{1}{\Gamma(\alpha)}, \frac{2}{\Gamma(\alpha-\mu)}\right\}$,

$$
\|Q_n x\| \leq K\left(q(1/n) + p(\|x\|) + p(1) + \omega(\|x\|)\right)\|\gamma\|_L
$$

for $x \in \mathcal{P}$. Since $\lim_{x\to\infty} \dfrac{p(x)+\omega(x)}{x} = 0$, there exists an $S > 0$ such that

$$K\left(q\left(1/n\right) + p(S) + p(1) + \omega(S)\right)\|\gamma\|_L < S.$$

Let $\Omega_2 = \{x \in X : \|x\| < S\}$. Then $\|Q_n x\| \leq \|x\|$ for $x \in \mathcal{P} \cap \partial\Omega_2$.

It follows from Theorem 3.2 that Q_n has a fixed point $x^* \in \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$. Consequently, (3.1), (1.2) has a solution x^* with $\|x^*\| \geq M$. □

4. POSITIVE SOLUTIONS OF THE SINGULAR PROBLEM

Lemma 4.1. *Let* (H1) *and* (H2) *hold. Let x_n be a solution to* (3.1), (1.2). *Then the sequences $\{x_n\}$ and $\{D^{\mu}_{0+}x_n\}$ are relatively compact in $C[0,1]$.*

Proof. Similar to the proof of Lemma 3.3, we use Arzelà-Ascoli to show these sequences are relatively compact. Note that

$$x_n(t) = \int_0^1 G(t,s) f_n\left(s, x_n(s), D^{\mu}_{0+}x_n(s)\right) ds$$

and

$$D^{\mu}_{0+}x_n(t) = \frac{1}{\Gamma(\alpha-\mu)}\left(t^{\alpha-\mu-1}\int_0^1 (1-s)^{\alpha-\beta-1} f_n\left(s, x_n(s), D^{\mu}_{0+}x_n(s)\right) ds \right.$$
$$\left. - \int_0^t (t-s)^{\alpha-\mu-1} f_n\left(s, x_n(s), D^{\mu}_{0+}x_n(s)\right) ds\right)$$

for $t \in [0,1]$ and $n \in \mathbb{N}$. It follows from the proof of Lemma 3.4 that $x_n(t) \geq Mt^{\alpha-1}$ for all $t \in [0,1]$, $n \in \mathbb{N}$. But

$$f_n\left(t, x_n(t), D^{\mu}_{0+}x_n(t)\right) \leq \gamma(t)\left(q(x_n(t)) + p(x_n(t)) + p(1) + \omega\left(\left|D^{\mu}_{0+}x_n(t)\right|\right)\right).$$

It was assumed that q is nonincreasing and p and ω are nondecreasing. Therefore,

$$f_n(t, x_n(t), D^{\mu}_{0+}x_n(t)) \leq \gamma(t)(q(Mt^{\alpha-1}) + p(\|x_n\|_0) + p(1) + \omega(\|D^{\mu}_{0+}x_n\|_0).$$

This implies

$$x_n(t) \leq \frac{1}{\Gamma(\alpha)}\left[\int_0^1 \gamma(t) q(Mt^{\alpha-1})dt + (p(\|x_n\|_0) + p(1) + \omega(\|D^{\mu}_{0+}x_n\|_0))\|\gamma\|_L\right],$$

and

$$D^{\mu}_{0+}x_n(t)$$
$$\leq \frac{2}{\Gamma(\alpha-\mu)}\left[\int_0^1 \gamma(t) q(Mt^{\alpha-1})dt + (p(\|x_n\|_0) + p(1) + \omega(\|D^{\mu}_{0+}x_n\|_0))\|\gamma\|_L\right],$$

for all $t \in [0,1]$ and $n \in \mathbb{N}$. Note it was assumed that $\int_0^1 \gamma(t)q(Mt^{\alpha-1})dt < \infty$. Therefore, by again setting $K = \max\left\{\frac{1}{\Gamma(\alpha)}, \frac{2}{\Gamma(\alpha-\mu)}\right\}$,

$$\|x_n\| \le K \left[\int_0^1 \gamma(t)q(Mt^{\alpha-1})dt + (p(\|x_n\|_0) + p(1) + \omega(\|Dux_n\|_0))\|\gamma\|_L \right],$$

for $n \in \mathbb{N}$. Since $\lim_{x\to\infty} \dfrac{p(x)+\omega(x)}{x} = 0$, there exists an $S > 0$ such that

$$K \left[\int_0^1 \gamma(t)q(Mt^{\alpha-1})dt + (p(v) + p(1) + \omega(v))\|\gamma\|_L \right] < S,$$

for each $v \ge S$. Thus $\|x_n\| < S$ for $n \in \mathbb{N}$ and the sequences $\{x_n\}$ and $\{D^{\mu}_{0+}x_n\}$ are uniformly bounded in $C[0,1]$.

Now, we show the sequences $\{x_n\}$ and $\{D^{\mu}_{0+}x_n\}$ are equicontinuous in $C[0,1]$. Let $0 \le t_1 < t_2 \le 1$. Using the fact that

$$0 < f_n(t, x_n(t), D^{\mu}_{0+}x_n(t)) \le \gamma(t)(q(Mt^{\alpha-1}) + p(S) + p(1) + \omega(S)),$$

we have

$$\begin{aligned}
&|x_n(t_2) - x_n(t_1)| \\
&\le \Gamma(\alpha) \Bigg((t_2^{\alpha-1} - t_1^{\alpha-1}) \int_0^1 (1-s)^{\alpha-1-\beta}(\gamma(s)(q(Ms^{\alpha-1}) + p(S) + p(1) + \omega(S)))ds \\
&\quad + \int_0^{t_1} \left((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha} - 1\right)(\gamma(s)(q(Ms^{\alpha-1}) + p(S) + p(1) + \omega(S)))ds \\
&\quad + (t_2-t_1)^{\alpha-1} \int_{t_1}^{t_2} (\gamma(s)(q(Ms^{\alpha-1}) + p(S) + p(1) + \omega(S)))ds \Bigg),
\end{aligned}$$

and

$$
\begin{aligned}
&|(D^{\mu}_{0^+}x_n)(t_2) - (D^{\mu}_{0^+}x_n)(t_1)| \\
&\le \frac{1}{\Gamma(\alpha-\mu)}\Bigg((t_2^{\alpha-\mu-1} - t_1^{\alpha-\mu-1})\times \\
&\int_0^1 (1-s)^{\alpha-\beta-1}(\gamma(s)(q(Ms^{\alpha-1}) + p(S) + p(1) + \omega(S)))ds \\
&\quad + \int_0^{t_1} \left((t_2-s)^{\alpha-\mu-1} - (t_1-s)^{\alpha-\mu-1}\right)(\gamma(s)(q(Ms^{\alpha-1}) + p(S) + p(1) + \omega(S)))ds \\
&\quad + (t_2-t_1)^{\alpha-\mu-1}\int_{t_1}^{t_2}(\gamma(s)(q(Ms^{\alpha-1}) + p(S) + p(1) + \omega(S)))ds \Bigg).
\end{aligned}
$$

Thus, with the appropriate choice of δ, it can be shown that for $\epsilon > 0$, if $t_2 - t_1 < \delta$, $|x_n(t_2) - x_n(t_1)| < \epsilon$ and $|(D^{\mu}_{0^+}x_n)(t_2) - (D^{\mu}_{0^+}x_n)(t_1)| < \epsilon$. Therefore, $\{x_n\}$ and $\{D^{\mu}_{0^+}x_n\}$ are equicontinuous in $C[0,1]$. So $\{x_n\}$ and $\{D^{\mu}_{0^+}x_n\}$ are relatively compact in $C[0,1]$. $\square$

Theorem 4.2. *Let* (H1) *and* (H2) *hold. Then* (1.1), (1.2) *has a positive solution* x *with* $x(t) \ge Mt^{\alpha-1}$ *for* $t \in [0,1]$.

Proof. From Lemma 3.4, (3.1), (1.2) has a positive solution for each $n \in \mathbb{N}$. Call these solutions x_n. From Lemma 4.1, the sequence $\{x_n\}$ is relatively compact in X. Therefore, without loss of generality, there exists an $x \in X$ with $\lim_{n\to\infty} x_n = x$ uniformly in X. Consequently, $x \in P$, $x(t) \ge Mt^{\alpha-1}$ for $t \in [0,1]$ and

$$\lim_{n\to\infty} f_n(t, x_n(t), D^{\mu}_{0^+}x_n(t)) = f(t, x(t), D^{\mu}_{0^+}x(t)),$$

for a.e. $t \in [0,1]$. Since

$$0 \le G(t,s)f_n(x_n(s), D^{\mu}_{0^+}x_n(s)) \le \frac{1}{\Gamma(\alpha)}\gamma(s)(q(Ms^{\alpha-1}) + p(S) + p(1) + \omega(S)) \in L^1[0,1]$$

for a.e. $s \in [0,1]$ and all $t \in [0,1]$, $n \in \mathbb{N}$, it follows from the Lebesgue Dominated Convergence Theorem that

$$\lim_{n\to\infty}\int_0^1 G(t,s)f_n(x_n(s), D^{\mu}_{0^+}x_n(s))ds = \int_0^1 G(t,s)f(t, x(t), D^{\mu}_{0^+}x(t))ds.$$

Since

$$x_n(t) = \int_0^1 G(t,s)f_n(s, x_n(s), D^{\mu}_{0^+}x_n(s))ds,$$

for $t \in [0,1]$,

$$x(t) = \int_0^1 G(t,s) f(t, x(t), D_{0+}^{\mu} x(t)) ds,$$

for $t \in [0,1]$. Thus, x is a positive solution of (1.1), (1.2). $\square$

5. EXAMPLE

Example 5.1. Fix $\alpha \in (1,2]$, $\beta \in (0, \alpha - 1]$, $\mu \in (0, \alpha - 1]$. Let $i, k \in (0,1)$, $j \in \left(0, \frac{1}{\alpha - 1}\right)$. Define

$$f(t,x,y) = \frac{1}{\sqrt{|2t-1|}} \left(x^i + \frac{1}{x^j} + |y|^k \right).$$

Additionally, set $\gamma(t) = \frac{1}{\sqrt{|2t-1|}}$, $q(x) = \frac{1}{x^j}$, $p(x) = x^i$, $\omega(y) = y^k$, $m = 1$ and $M = \frac{\beta}{(\alpha-\beta)\Gamma(\alpha+1)}$.

Notice that for $t \in [0,1] \setminus \{\frac{1}{2}\}$ and $(x,y) \in (0,\infty) \times \mathbb{R}$,

$$f(t,x,y) \geq \frac{1}{\sqrt{|2t-1|}} \geq 1 = m.$$

Hence f satisfies condition (H1). Also, $f(t,x,y) = \gamma(t)(q(x) + p(x) + \omega(|y|)$, $\gamma \in L^1[0,1]$, $q \in C(0,\infty)$ is nonincreasing, and $p, \omega \in C[0,\infty)$ are nondecreasing. Last,

$$\int_0^1 \frac{M^{-j} t^{-j(\alpha-1)}}{\sqrt{|2t-1|}} dt < \infty,$$

since $j(\alpha - 1) < 1$, and

$$\lim_{x \to \infty} \frac{x^i + x^k}{x} = 0,$$

since $i, k \in (0,1)$. So (H2) is also satisfied. Thus, Theorem 4.2 provides that there is at least one positive solution $x(t)$ to the fractional differential equation

$$D_{0+}^{\alpha} x + \frac{1}{\sqrt{|2t-1|}} \left(x^i + \frac{1}{x^j} + |D_{0+}^{\mu} x|^k \right) = 0,$$

satisfying

$$x(0) = D_{0+}^{\beta} x(1) = 0.$$

Further, for $t \in [0,1]$,

$$x(t) \geq \frac{\beta t^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha+1)}.$$

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