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Gaussian Amicable Pairs

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AMICABLE GAUSSIAN INTEGERS

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DEDICATION

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Abstract

Amicable pairs are two integers where the sum of the proper divisors of one is the other and vice versa. Since the Gaussian integers have many of the properties of the regular integers, we sought to discover whether there exist any pairs of Gaussian integers with the same property. It turns out that they do exist. In fact, some of the normal amicable pairs carry over as Gaussian amicable pairs. Also discovered are pairs that have a complex part.

Contents

Part 1

Introduction

Sum of Divisors Function

The sum of divisors function $\sigma(n)$ for a number n denotes the sum of the positive factors of n where $\sigma(n) = \sum_{d|n} d$. For example the positive divisors of 24 are $1, 2, 3, 4, 6, 8, 12,$ and 24, hence $\sigma(24) = 1+2+3+4+6+8+12+24 = 60$. This method of calculating $\sigma(n)$ will become less efficient the larger n becomes. (Imagine trying to list and then sum all of the proper divisors of $456,892$!) In order to work more conveniently with $\sigma(n)$ we must first note that $\sigma(n)$ is a multiplicative function. That is, $\sigma(mn) = \sigma(m)\sigma(n)$, where m and n are relatively prime. Since distinct primes raised to different powers are relatively prime, if $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_s^{\alpha_s}$ then $\sigma(n) = \sigma(p_1^{\alpha_1})\sigma(p_2^{\alpha_2}) \cdot \ldots \cdot \sigma(p_s^{\alpha_s})$. Consider again $\sigma(24)$. This property tells us that $\sigma(24) = \sigma(2^3 \cdot 3) = \sigma(2^3)\sigma(3)$. But how do we calculate the values of $\sigma(2^3)$ and $\sigma(3)$ without simply listing the proper divisors of each number?

Theorem 8.7 from [3] notes that if p is any prime and e is any positive integer, then $\sigma(p) = p+1$ and $\sigma(p^e) = \dfrac{p^{e+1}-1}{p-1}$. Using these properties and multiplicity we can evaluate:

$$\begin{aligned}
\sigma(24) &= \sigma(2^3)\sigma(3) \\
&= \left(\frac{2^{3+1}-1}{2-1}\right)(3+1) \\
&= (15)(4) \\
&= 60
\end{aligned}$$

There are many interesting numerical properties that stem from the sum of divisors function. For example, if n is a number and $\sigma(n) = 2n$ then n is called a *perfect* number. Perfect numbers are interesting in their own right, and they are also extremely significant in the study of Mersenne primes. Mersenne primes are prime numbers of the form $2^p - 1$ where

p is prime. In fact if $2^p - 1$ is prime, then $(2^p - 1)(2^{p-1})$ is a perfect number. Amicable pairs define another relationship that stems from the sum of divisors function. Two integers m and n are said to be *amicable* if $\sigma(m) - m = n$ and $\sigma(n) - n = m$. So two integers m and n are amicable if the sum of proper divisors of m is n and the sum of the proper divisors of n is m. The pair (m, n) is called an *amicable pair*. This concept will be the primary focus of this paper.

Amicable Pairs in the Integers

Amicable pairs are also referred to as *friendly* numbers. The first amicable pair, $(220, 284)$, is credited to have been discovered by the mathematician Pythagoras who lived around 600 B.C. His Pythagorean brotherhood believed that amicable pairs had mysterious powers, and regarded these numbers as signs of friendship [2]. Amicable pairs have been noted in scripture, particularly in Genesis when Jacob gave Esau 220 goats as a sign of love and comradery. There are others, such as John Conway, who are not as convinced in the usefulness of amicable pairs. Conway has been quoted, saying,"The only application or use for these numbers is the original one – you insert a pair of amicable numbers into a pair of amulets, of which you wear one yourself and give the other to your beloved!"

Despite these sentiments, amicable pairs have been studied for centuries by a plethora of mathematicians. Though Pythagoras discovered the smallest pair, mathematicians like Euler, Fermat, Descartes, and Thābit ibn Qurra helped pave the way for modern researchers of amicable pairs. Determining methods for generating amicable pairs focuses explicitly on concepts of primality and manipulations of the sum of divisors function. Before going into more depth on some methods for finding pairs, let us first demonstrate how to use the sum of divisors function to determine if two numbers are amicable.

As noted above, the smallest amicable pair in the integers is the pair $(220, 284)$. To illustrate this, let $m = 220$ and $n = 284$. Then

$$\begin{aligned}
\sigma(220) &= \sigma(2^2 \cdot 5 \cdot 11) \\
&= \sigma(2^2)\sigma(5)\sigma(11) \\
&= \left(\frac{2^3 - 1}{2 - 1}\right)(5 + 1)(11 + 1) \\
&= (7)(6)(12) \\
&= 504,
\end{aligned}$$

and

$$\begin{aligned}
\sigma(m) - m &= \sigma(220) - 220 \\
&= 504 - 220 \\
&= 284 \\
&= n.
\end{aligned}$$

Similary

$$\begin{aligned}
\sigma(284) &= \sigma(2^2 \cdot 71) \\
&= \sigma(2^2)\sigma(71) \\
&= \left(\frac{2^3 - 1}{2 - 1}\right)(71 + 1) \\
&= (7)(72) \\
&= 504,
\end{aligned}$$

and

$$\begin{aligned}\sigma(n) - n &= \sigma(284) - 284 \\ &= 504 - 284 \\ &= 220 \\ &= m.\end{aligned}$$

Hence we see that 220 and 284 are amicable.

Amicable pairs are often organized by type. In the example above, we see that $220 = 2^2 \cdot 5 \cdot 11$ and $284 = 2^2 \cdot 71$. They have a common factor of 2^2 and 220 has two separate distinct primes in its factorization whereas 284 has one other distinct prime in its factorization. Because of this, pairs like $(220, 284)$ are called $(2, 1)$ pairs. In the pair $(10744, 10856)$, discovered by Euler in 1747, $10744 = 2^3 \cdot 17 \cdot 79$ and $10856 = 2^3 \cdot 23 \cdot 59$. Hence $(10744, 10856)$ is a $(2, 2)$ pair [4]. There are some cases where two numbers in an amicable pair will have a common factor raised to different powers. For example in the pair $(79750, 88730)$, $79750 = 2 \cdot 5^3 \cdot 11 \cdot 29$ and $88730 = 2 \cdot 5 \cdot 19 \cdot 467$. Five is raised to a larger power in the first number than it is in the second. In cases like these, we call $(79750, 88730)$ an *exotic* pair.

Many of the methods to generate amicable pairs deal with pairs of the type $(2, 1)$. Thābit ibn Qurra was a 9^{th} century astronomer who discovered a method for finding amicable pairs. He proved that if

$$\begin{aligned}&p = 3 \cdot 2^{n-1} - 1, \\ &q = 3 \cdot 2^n - 1, \\ &\text{and} \\ &r = 9 \cdot 2^{2n-1} - 1\end{aligned}$$

are all prime numbers where $n > 1$, then $(2^n pq, 2^n r)$ are amicable. If $n = 2$ then we get $(2^2 \cdot 5 \cdot 11, 2^2 \cdot 71) = (220, 284)$. Numbers of the form $3 \cdot 2^n - 1$ are hence called Thābit numbers and are sometimes referred to as $3 - 2 - 1$ numbers. When a value of n yields a prime number then these are called Thābit primes [2].

Euler also had a method for generating amicable pairs of the type $(2, 1)$. He proved that if

$$\begin{aligned} p &= (2^{n-m} + 1)(2^m - 1), \\ q &= (2^{n-m} + 1)(2^n - 1), \\ &\text{and} \\ r &= (2^{n-m} + 1)^2(2^{m+n} - 1) \end{aligned}$$

are all prime numbers and $n > m > 0$ then $(2^n pq, 2^n r)$ are amicable [8]. His method is considered to be a generalization of Thābit's method. Over the years, many mathematicians have adopted methods for generating amicable pairs that stem from the same ideas used by mathematicians centuries ago. We will discuss some of these modern methods in later sections.

Gaussian Integers

The set of *Gaussian integers*, denoted $\mathbb{Z}_i$, is given by $\mathbb{Z}_i = \{a + bi \mid a, b \in \mathbb{Z}\}$. The $\mathbb{Z}_i$ form a ring, and addition and multiplication are defined below:

$$\begin{aligned} (a + bi) + (c + di) &= (a + c) + (b + d)i \\ (a + bi) \times (c + di) &= (ac - bd) + (ad + bc)i. \end{aligned}$$

An important property in $\mathbb{Z}_i$ is that every $z \in \mathbb{Z}_i$ can be represented geometrically as a vector in the complex plane. The length of this vector is called the *magnitude* of a complex number $z = a + bi$, denoted $|z|$, where

$$|z| = |a + bi| = \sqrt{a^2 + b^2}.$$

The concept of magnitude is very useful when comparing two Gaussian integers. An even more useful concept is the square of the magnitude, called the norm. The *norm* of a complex number $z = a+bi$, denoted $N(z)$, is defined

$$N(z) = a^2 + b^2.$$

One important property of the norm of a Gaussian integer is that it is completely multiplicative. That is, if $r, s \in \mathbb{Z}_i$, then:

$$N(rs) = N(r)N(s)$$

There are some texts that do not distinguish between the magnitude and the norm. In this paper we will use the established conventions mentioned above.

The norm of a Gaussian integer is important when classifying Gaussian primes. In the integers, a number p is prime if its only positive divisors are p and 1. In order to define a Gaussian prime, we first must define a unit in $\mathbb{Z}_i$. If $\epsilon \in \mathbb{Z}_i$ then we say ϵ is a *unit* if there exists $z \in \mathbb{Z}_i$ such that $\epsilon \cdot z = 1$. So units are elements in the Gaussian integers whose multiplicative inverses in $\mathbb{C}$ are also in the Gaussian integers. There are only four elements in $\mathbb{Z}_i$ that meet this criteria. The set $\{1, -1, i, -i\}$ comprises all four units in the Gaussian integers. Hence, if $N(z) = 1$ where $z \in \mathbb{Z}_i$, then z must be a unit. Now we can define a prime p in $\mathbb{Z}_i$.

Suppose $p \in \mathbb{Z}_i$ where p is not a unit. Then p is *prime* if for every $a, b \in \mathbb{Z}_i$, $p = ab$ implies that either a or b is a unit. Let's consider 2, the first prime in the integers. In $\mathbb{Z}_i$, $2 = (1+i)(1-i)$, yet neither $(1+i)$ or $(1-i)$ is a unit. Hence 2 is not prime in $\mathbb{Z}_i$. To see this illustrated more clearly, consider:

$$N(2) = 2^2 = 4$$

and suppose that $2 = ab$, where $a, b \in \mathbb{Z}_i$. Then because the norm is completely multiplicative,

$$\begin{aligned} N(2) &= N(a)N(b) \\ 4 &= N(a)N(b) \end{aligned}$$

So we must have the $N(a) = 2$ and $N(b) = 2$. Since $N(1+i) = 1^2 + 1^2 = 2$ and $N(1-i) = 1^2 + (-1)^2 = 2$ we see that neither of these norms is equivalent to a unit in $\mathbb{Z}_i$ and so 2 is not prime in $\mathbb{Z}_i$.

Let's consider a similar calculation to determine if $2 + 3i$ is prime as a Gaussian integer.

$$\begin{aligned} N(2+3i) &= 2^2 + 3^2 \\ &= 13 \end{aligned}$$

and suppose that $13 = ab$, where $a, b \in \mathbb{Z}_i$. Then because the norm is completely multiplicative,

$$\begin{aligned} N(2+3i) &= N(a)N(b) \\ 13 &= N(a)N(b) \end{aligned}$$

Since 13 is prime in the integers, it must be that either $N(a) = 1$ or $N(b) = 1$. In either case, a or b is a unit and it follows that $2 + 3i$ is prime as a Gaussian integer.

This last example illustrates a very important fact. That is, if z is a Gaussian integer, and the $N(z) = p$ where p is prime in $\mathbb{Z}$, then z is prime in $\mathbb{Z}_i$ [1]. Because of this, calculating the norms of elements in $\mathbb{Z}_i$ is an essential tool in determining the primes p in $\mathbb{Z}$ that are also prime in $\mathbb{Z}_i$. It is also an essential tool in determining the primality of Gaussian integers, and hence of factoring Gaussian integers into the products of prime factors.

We have already shown that 2 is not prime in $\mathbb{Z}_i$, but what about $3, 5, 7, \ldots$? We can show that 3 is prime in $\mathbb{Z}_i$. First we suppose that 3 is not prime in $\mathbb{Z}_i$. Then there are elements $c, d \in \mathbb{Z}_i$ such that $3 = ab$ where a is not a unit and b is not a unit. Then we have,

$$\begin{aligned} N(3) &= N(a)N(b) \\ 9 &= N(a)N(b). \end{aligned}$$

So it must be that $N(a) = N(b) = 3$ since we assumed that a, b are not units. Then we can write a as a Gaussian integer where $a = c + di$ and

$$\begin{aligned} N(a) &= c^2 + d^2 \\ 3 &= c^2 + d^2. \end{aligned}$$

Simplifying we have,

$$c^2 = 3 - d^2$$

and so

$$c = \pm\sqrt{3 - d^2}.$$

There are no integer values for d in the above equation that will give an integer value for c, and so there are no integer values of c and d such that

$3 = c^2 + d^2$. Hence we arrive at a contradiction and it follows that 3 must be prime in $\mathbb{Z}_i$.

Aside from 2, every prime integer p is odd and is thus of the form $4k+1$ or $4k+3$. In the example above, 3 is obviously of the form $4k+3$ and it follows that any prime p in $\mathbb{Z}$ that is of the form $4k+3$, also cannot be written as the sum a^2+b^2 where $a, b \in \mathbb{Z}$. This leads to the very important fact that if a prime $p \in \mathbb{Z}$ can be written in the form $4k+3$, then p is prime in $\mathbb{Z}_i$ [1]. Hence the primes $3, 7, 11, 19,$ etc. in the integers are also primes in the Gaussian integers.

Now we consider primes in the integers of the form $4k+1$. Fermat was able to prove that if p is an odd prime of the form $4k+1$ then p can be written as the sum a^2+b^2 where $a, b \in \mathbb{Z}$. So if an integer p is of the form $4k+1$ it can be written as a sum of squares [1]. It follows that such a p in the integers will not be prime in the Gaussian integers. Instead it will breakdown in $\mathbb{Z}_i$ where $p = (a+bi)(b+ai)\epsilon$ where ϵ is a unit. Consider the integer 5, then since 5 is of the form $4k+1$ and is equivalent to 1^2+2^2 in $\mathbb{Z}$, we have $5 = (1+2i)(2+i)(-i)$ in $\mathbb{Z}_i$.

Considering all of the above, it is possible to concisely characterize Gaussian Primes in the following manner. Let $r \in \mathbb{Z}_i$ where $r = a+bi$ and $a, b \in \mathbb{Z}$. Then,

<u>Case 1: $a \neq 0$ and $b = 0$</u>

- If a is composite then r is not a Gaussian prime
- If a is prime of the form $4k+1$ then r is not a Gaussian prime
- If a is prime of the form $4k+3$ then r is a Gaussian prime

<u>Case 2: $a = 0$ and $b \neq 0$</u>

- If the $N(r) = c$ where c is composite in $\mathbb{Z}$ then r is not a Gaussian prime
- If the $N(r) = p$ where p is prime in $\mathbb{Z}$ then r is a Gaussian prime

Case 3: $a \neq 0$ and $b \neq 0$

- If the $N(r) = c$ where c is composite in $\mathbb{Z}$ then r is not a Gaussian prime
- If the $N(r) = p$ where p is prime in $\mathbb{Z}$ then r is a Gaussian prime

Hence, given any element in $\mathbb{Z}_i$ it is possibly to quickly determine whether or not it is prime. In later sections this idea will be expanded upon to find a way to factor Gaussian integers. First we must highlight an important formula given in the following section.

Complex Sum of Divisors Function

Let η be a Gaussian integer such that $\eta = \epsilon \prod \pi_i^{k_i}$ where ϵ is a unit and each π_i lies in the first quadrant. Then we define the complex sum of divisors function as follows:

$$\sigma^\star(\eta) = \prod \frac{\pi_i^{k_i+1} - 1}{\pi_i - 1}$$

This definition of the complex sum of divisors function ensures that $\sigma^\star$ is multiplicative and satisfies the necessary condition that $|\sigma^\star(\eta)| \geq |\eta|$. There exist analogous definitions where each prime π_i lies in the fourth quadrant or in some combination of the first and fourth quadrants. The above definition was discovered by Robert Spira of Berkley and was chosen as a matter of convenience so that every associate of the Gaussian integer η would have positive integer coefficients [5].

The complex sum of divisors function is an extension of the real-valued sum of divisors function. However, where $\sigma(24) = 60$, the complex $\sigma^\star$-function gives $\sigma^\star(24) = -32 - 28i$. This has to do with the different factorizations of 24 in $\mathbb{Z}$ and $\mathbb{Z}_i$. In $\mathbb{Z}_i$, $24 = i(1+i)^6(3)$. So using the definition above we see:

$$
\begin{aligned}
\sigma^{\star}(24) &= \left(\frac{(1+i)^{6+1}-1}{1+i-1}\right)(3+1) \\
&= (-8-7i)(4) \\
&= -32-28i
\end{aligned}
$$

The different factorization of numbers in $\mathbb{Z}$ and $\mathbb{Z}_i$ will be addressed in later sections.

Similar to the real-valued σ function, there are several numerical properties that stem from the complex $\sigma^{\star}$-function. We can say a complex numbers η is perfect if $\sigma^{\star}(\eta) = 2\eta$, or more specifically if $\sigma^{\star}(\eta) = (1+i)\eta$. As noted previously, perfect numbers are interlinked with Mersenne primes. Complex Mersenne primes are of the form $M_p = \dfrac{\pi^p - 1}{\pi - 1}$ where $\pi = 1+i$ and p is a rational prime. If M_p is a complex Mersenne prime where p is of the form $8k+1$, then $(1+i)^{p-1} \cdot M_p$ is a perfect number [5]. Another natural extension of the complex sum of divisors function is to investigate amicable pairs in the complex plane. For the sake of this paper, we will henceforth focus on amicable pairs in the Gaussian integers.

Amicable Pairs in the Gaussian Integers

Similar to the definition of an amicable pair in the integers, amicable pairs in the Gaussian integers can be identified using the complex sum of divisors function. Two Gaussian integers m and n are said to be amicable if $\sigma^{\star}(m) - m = n$ and $\sigma^{\star}(n) - n = m$. As noted previously, in order to calculate $\sigma^{\star}(\eta)$ where $\eta \in \mathbb{Z}_i$, then we must first factor η into its unique factorization up to order and units so that all of the factors of η lie in the first quadrant.

This process in itself tends to be a lot of more tedious than factoring large numbers into the product of primes in the integers. In later sections we will identity an algorithm to factor Gaussian integers in the manner described above so that we can use the complex $\sigma^{\star}$-function to find amicable pairs in $\mathbb{Z}_i$. Also, we will answer the question if there exist any amicable pairs in the integers that are also amicable in the Gaussian integers. In addition to this question we will also explore amicable pairs in the Gaussian integers.

Part 2

Factoring Gaussian Integers

Factoring Algorithm for Gaussian Integers

In order to apply the complex sum of divisors function $\sigma^\star$ to determine whether or not a pair in the integers is amicable in the Gaussian integers, or if two Gaussian integers are amicable, we must first be able to factor numbers efficiently. Consider for example the Gaussian integer $-46 + 20i$. How do we go about trying to factor this in $\mathbb{Z}_i$? It turns out that there are several different factorizations. For example:

$$\begin{aligned} -46 + 20i &= (1+i)^2(1+4i)(1+6i)(-i) \\ &= (1+i)(1-i)(1+4i)(1+6i) \\ &= (1+i)^2(4-i)(1+6i) \\ &= (1+i)^2(1+4i)(6-i) \\ &= (1+i)^2(-4+i)(1+6i)(-1) \end{aligned}$$

and so on. But recall that the function $\sigma^\star$ requires that every number be broken down so that every factor lies in the first quadrant apart from units in $\mathbb{Z}_i$. So in this example above, the first factorization given for $-46 + 20i$ is the factorization required to use $\sigma^\star$.

In order to factor the above example, instead of arbitrarily trying to divide away Gaussian integers, it is most efficient to find the norm first. When we do this we find that $N(-46+20i) = 2516$. We then find the prime factorization of 2516, that is, $2516 = 2^2 \cdot 17 \cdot 37$. Here we see the norm of this Gaussian integer contains a power of 2 as well as two elements in $\mathbb{Z}_i$ that are of the form $4k+1$. This means that there exist Gaussian integers $a+bi$ and $c+di$ such that $N(a+bi) = 17$ and $N(c+di) = 37$. In each case there are 8 possibilities for both $a+bi$ and $c+di$. For example $a+bi$ could be any members of the set $\{1+4i, 1-4i, -1-4i, -1+4i, 4+i, 4-i, -4-i, -4+i\}$. However, because $\sigma^\star$ requires that each factor lie in the first quadrant we can eliminate these eight choices to just $1+4i$ or $4+i$. Similarly, we can eliminate all possibilities of $c + di$ to $1 + 6i$ or $6 + i$. Hence determining a factorization of $-46 + 20i$ where each factor lies in the first quadrant is reduced to testing the divisibility of two factors whose norms are of the form $4k + 1$ for each prime p in the factorization of the norm that meets this criteria. These ideas led to the development of a factoring algorithm for elements in $\mathbb{Z}_i$ so that any Gaussian integer could be factored efficiently in such a way that every factor lies in the first quadrant outside of units.

FACTORING ALGORITHM FOR GAUSSIAN INTEGERS

Step 1

Compute the norm (denoted N) of the Gaussian integer $a + bi$ where $N(a + bi) = a^2 + b^2$. Factor this integer into its distinct prime factorization $p_1^{n_1} \cdot p_2^{n_2} \cdot \ldots \cdot p_s^{n_s}$ in $\mathbb{Z}$

Step 2

For each p_i identify whether $p_i = 2$ or if p_i is of the form $4k + 1$ or of the form $4k + 3$.

Step 3

a.) For each p_i of the form $4k+3$, the exponent n_i should be even, that is $n_i = 2m_i$ where $m_i \in \mathbb{Z}$. Then $p_i^{m_i}$ is a factor of $a+bi$. Write down this factor.

b.) For each $p = p_i$ of the form $4k+1$, this p can be written as a Gaussian integer $c+di$ where $N(c+di) = p$. Use PowerRepresentations[p,2,2] in Mathematica® or a similar program to find c and d. Do this for each p_i of the form $4k+1$ and store each $(c_i, d_i)[n_i]$ where n_i is the exponent on each p_i.

c. If $p_i = 2$ and m_i is the exponent on p_i, then $(1+i)^{m_i}$ is a factor of $a+bi$. Write down this factor.

Step 4

If there are no p_i of the form $4k+3$ or $p_i = 2$ let $x+yi = a+bi$ and continue to **Step 5.** If there are, divide $a+bi$ by each factor p_i and $(1+i)^{m_i}$ identified in **Step 2**. Let this result equal $x+yi$.

Step 5

i.) For each p_i of the form $4k+1$ consider the pairs $(c_i, d_i)[n_i]$ found in **Step 3b)**. Starting with $(c,d)[n] = (c_1, d_1)[n_1]$ divide $x+yi$ by $(c+di)^n$. If the result is a Gaussian integer, write $(c+di)^n$ down as a factor of $a+bi$. Repeat for the next $(c_i, d_i)[n_i]$. Stop when $x+yi = \{1, -1, i, -i\}$. Otherwise, reduce n by 1 and divide $x+yi$ by $(c+d)^{n-1}$. Continue this until the result is a Gaussian integer or $n = 1$.

If the result is a Gaussian integer, let k be the final reduced power of n and write down $(c+di)^k$ as a factor of $a+bi$. Let the resulting Gaussian integer $= x+yi$. Then go to **ii.)**

If the result is not a Gaussian integer then $n = 1$ and go to **ii**.

ii.) If the step above resulted in a Gaussian Integer let $m = n_i - k$. If not let $m = n_i$. In each case n_i is the original exponent coupled with the corresponding (c_i, d_i). Divide $x + yi$ by $(d + ci)^m$. The result should be a Gaussian integer. Write down $(d + ci)^m$ as a factor. Let the resulting Gaussian integer $= x + yi$. Then go back to **i.)** for the next $(c_i, d_i)[n_i]$ and repeat until $x + yi = \{1, -1, i, -i\}$.

Examples Using the Factoring Algorithm

Now we provide two examples on how to factor Gaussian integers using the factoring algorithm outlined in the previous section.

Example 1 Factoring: $-3235 + 1020i$

Step 1

$N(-3235 + 1020i) = (-3235)^2 + (1020)^2 = 11505625$

$11505625 = 5^4 \cdot 41 \cdot 449$

Step 2

$5, 41,$ and 449 are all of the form $4k + 1$.

Step 3

b.)

$$\begin{cases} 5^4 \to (c_1, d_1)[n_1] = (1, 2)[4] \\ 41 \to (c_2, d_2)[n_2] = (4, 5)[1] \\ 449 \to (c_3, d_3)[n_3] = (7, 20)[1] \end{cases}$$

Step 4

Since there are no p_i of the form $4k+3$ let $x+yi=-3235+1020i$ and continue to **Step 5**.

<u>**Step 5**</u>

i. Start with $(c_1,d_1)[n_1]=(1,2)[4]$ and divide $-3235+1020i$ by $(1+2i)^4$.

$$\frac{-3235+1020i}{(1+2i)^4}=\frac{-367}{125}-\frac{16956}{125}i$$

This is not a Gaussian integer. Reduce n_1 by 1.

$$\frac{-3235+1020i}{(1+2i)^3}=\frac{6709}{25}-\frac{3538}{25}i$$

This is not a Gaussian integer. Reduce the power by 1.

$$\frac{-3235+1020i}{(1+2i)^2}=\frac{2757}{5}+\frac{1976}{5}i$$

This is not a Gaussian integer. Reduce the power by 1.

$$\frac{-3235+1020i}{1+2i}=-239+1498i$$

This is a Gaussian integer. So $k=1$ and $1+2i$ is a factor of $-3235+1020i$. Let $x+yi=-239+1498i$ and go to **ii**.

ii. Let $m=4-1=3$. Now we divide $-3235+1020i$ by $(2+i)^3$.

$$\frac{239+1498i}{(2+i)^3}=128+45i.$$

This is a Gaussian integer and $(2+i)^3$ is a factor of $-3235+1020i$. Let $x+yi=128+45i$ and go back to **i.**

i. Now we take $(c_2,d_2)[n_2]=(4,5)[1]$ and divide $128+45i$ by $4+5i$.

$$\frac{128+45i}{4+5i}=\frac{737}{41}-46041i$$

This is not a Gaussian integer and $n_2 = 1$ so we go to **ii.**

ii. Let $m = n_2 = 1$ and divide $128 + 45i$ by $5 + 4i$.

$$\frac{128 + 45i}{5 + 4i} = 20 - 7i$$

This is a Gaussian integer and $5 + 4i$ is a factor of $-3235 + 1020i$. Let $x + yi = 20 - 7i$ and go back to **i**.

i. Now we take $(c_3, d_3)[n_3] = (7, 20)[1]$ and divide $20 - 7i$ by $7 + 20i$.

$$\frac{20 - 7i}{7 + 20i} = -i$$

This is a Gaussian integer and $7 + 20i$ is a factor of $-3235 + 1020i$. Then $x + yi = -i$ is the final factor and we have:

$$-3235 + 1020i = (1 + 2i)(2 + i)^3(5 + 4i)(7 + 20i)(-i)$$

Example 2 Factoring: $736 - 16560i$

Step 1

$N(736 - 16560i) = (736)^2 + (-16560)^2 = 274775296$

$274775296 = 2^8 \cdot 23^2 \cdot 2029$

Step 2

We have $p = 2$.

2029 is of the form $4k + 1$.

23 is of the form $4k + 3$.

Step 3

a.) The exponent on 23 is 2 where $2 = 2 \cdot 1$. So $m_1 = 1$ and 23 is a factor of $736 - 16560i$.

b.)

$$\begin{cases} 2029 \to (c_1, d_1)[n_1] = (2, 45)[1] \end{cases}$$

c.) We have $p = 2$ so $l = 8$ and $(1 + i)^8$ is a factor of $736 - 16560i$.

Step 4

Divide $736 - 16560i$ by $(1 + i)^8$ and then that result by 23.

$$\frac{736 - 16560i}{(1 + i)^8} = 46 - 1035i$$

$$\frac{46 - 1035i}{23} = 2 - 45i$$

Let $x + yi = 2 - 45i$ and go to **Step 5**.

Step 5

i. Start with $(c_1, d_1)[n_1] = (2, 45)[1]$ and divide $2 - 45i$ by $2 + 45i$.

$$\frac{-2021}{2029} - 1802029i$$

This is not a Gaussian integer and $n_1 = 1$ so we go to **ii.**

ii. Let $m = n_1 = 1$ and divide $2 - 45i$ by $45 + 2i$.

$$\frac{2 - 45i}{45 + 2i} = -i$$

This is a Gaussian integer and $45 + 2i$ is a factor of $736 - 16560i$. Then $x + yi = -i$ is the final factor and we have:

$$736 - 16560i = (1 + i)^8(45 + 2i)(23)(-i)$$

Part 3

Pairs that Carry Over

Certain (2,1) Pairs

The first question of interest is whether or not there exist amicable pairs in the integers that are also amicable in the Gaussian integers. If so, what properties must the members of such a pair have? Consider the smallest amicable pair mentioned earlier $(220, 284)$. In the integers $220 = 2^2 \cdot 5 \cdot 11$ and $284 = 2^2 \cdot 71$. However in the Gaussian integers $220 = (1+i)^4(1+2i)(2+i)(11)(i)$ and $284 = (1+i)^4(71)(-1)$. Applying the complex sum of divisors function we have,

$$\sigma^{\star}(220) = -672 - 144i$$

and

$$\sigma^{\star}(284) = -288 + 360i.$$

Thus

$$\sigma^{\star}(220) - 220 = 452 - 144i$$

and

$$\sigma^{\star}(284) - 284 = -572 - 360i$$

Hence the smallest amicable pair in $\mathbb{Z}$ does not carry over to $\mathbb{Z}_i$.

An important reason that this pair does not carry over has to do with the different factorization of 220 and 284 in the $\mathbb{Z}_i$. Because of this when we applied the complex sum of divisors function $\sigma^{\star}$ to both 220 and 284 we end up with an imaginary part. Note that the pair $(220, 284)$ is of the form $(2^2pq, 2^2r)$. We will prove that in fact there are *no* pairs in $\mathbb{Z}$ of the form $(2^npq, 2^nr)$ that are amicable in $\mathbb{Z}_i$. This is because for pairs of this form, when we apply $\sigma^{\star}$ to the second number in the pair we will always end up with an imaginary part. Before we prove this however, we need the following Theorem.

Theorem 1. *Let $\sigma^\star$ denote the complex sum of divisors function. Let n be an integer greater than or equal to 1. Then*

$$\sigma^\star(2^n) = (-1)^{\binom{n+4}{2}}2^n + (-1)^{\binom{n+3}{2}}(2^n + (-1)^{\binom{n+3}{2}})i$$

The actual proof of the formula from Theorem 1 will appear in Part IV of this paper. For now, note that this theorem implies in general that $\sigma^\star(2^n) = x + yi$ with $y \neq 0$, and specifically that $x = (-1)^{\binom{n+4}{2}}2^n$ and $y = (-1)^{\binom{n+3}{2}}(2^n+(-1)^{\binom{n+3}{2}})$. Since $\sigma^\star$ is multiplicative it will be important to have a formula that tells us how to calculate or even generalize $\sigma^\star(2^n)$. Lastly, before we begin the aforementioned proof, we must highlight a relationship between p, q and r in amicable pairs. Consider again a pair of the form $(2^n pq, 2^n r)$. Since they are amicable we know

$$\sigma(2^n pq) = \sigma(2^n r).$$

Then,

$$\begin{aligned} \sigma(2^n pq) &= \sigma(2^n r) \\ \sigma(2^n)\sigma(p)\sigma(q) &= \sigma(2^n)\sigma(r) \\ (p+1)(q+1) &= (r+1) \end{aligned}$$

After distributing and simplifying we are left with

$$r = pq + p + q.$$

Using Theorem 1 and the above relationship between p, q, and r we can now prove the following theorem.

Theorem 2. *There are no* $(2, 1)$ *pairs of the form* $(2^n pq, 2^n r)$ *in* $\mathbb{Z}$ *that are also amicable in* $\mathbb{Z}_i$ *where* $n, p, q, r \in \mathbb{Z}$.

Proof:

Case 1: Prove $\sigma^\star(2^a r) - 2^a r = c + di$ with $d \neq 0$ $\forall a, p, q \in \mathbb{Z}$ where p and q are of the form $4k + 3$.

Let $p = 4k + 3$ and $q = 4l + 3$ for some $k, l \in \mathbb{N}$ Then we have,

$$\begin{aligned} r &= pq + p + q \\ &= (4k + 3)(4l + 3) + (4k + 3) + (4l + 3) \\ &= 16kl + 12k + 12l + 9 + 4k + 3 + 4l + 3 \\ &= 16kl + 16k + 16l + 15 \\ &= 4(4kl + 4k + 4l + 3) + 3 \\ &= 4m + 3. \end{aligned}$$

where $m = 4kl + 4k + 4l + 3$. So r is of the form $4h + 3$. Using Theorem 1, for some $x, y \in \mathbb{Z}$ where $y \neq 0$,

$$\begin{aligned} \sigma^\star(2^a r) - 2^a r &= \sigma^\star(2^a)\sigma^\star(r) - 2^a r \\ &= (x + yi)(r + 1) - 2^a r \\ &= (x(r + 1) - 2^a r) + y(r + 1)i \\ &= c + di. \end{aligned}$$

where $c = x(r + 1) - 2^a r$ and $d = y(r + 1)$. Since $y \neq 0$, then $d \neq 0$ so $c + di$ has a non-zero imaginary part as desired.

Case 2: Prove $\sigma^\star(2^a r) - 2^a r = c + di$ with $d \neq 0$ $\forall a, p, q \in \mathbb{Z}$ where p and q are of the form $4k+1$.

Let $p = 4k+1$ and $q = 4l+1$ for some $k, l \in \mathbb{N}$. Then we have,

$$\begin{aligned}
r &= pq + p + q \\
&= (4k+1)(4l+1) + (4k+1) + (4l+1) \\
&= 16kl + 4k + 4l + 1 + 4k + 1 + 4l + 1 \\
&= 16kl + 8k + 8l + 3 \\
&= 4(4kl + 2k + 2l) + 3 \\
&= 4m + 3.
\end{aligned}$$

where $m = 4kl + 2k + 2l$. So we see r is of the form $4h+3$. Using Theorem 1, for some $x, y \in \mathbb{Z}$ where $y \neq 0$,

$$\begin{aligned}
\sigma^\star(2^a r) - 2^a r &= \sigma^\star(2^a)\sigma^\star(r) - 2^a r \\
&= (x + yi)(r+1) - 2^a r \\
&= (x(r+1) - 2^a r) + y(r+1)i \\
&= c + di.
\end{aligned}$$

where $c = x(r+1) - 2^a r$ and $d = y(r+1)$. Since $y \neq 0$, then $d \neq 0$ so $c + di$ has a non-zero imaginary part as desired.

Case 3: Prove $\sigma^{\star}(2^a r) - 2^a r = c + di$ with $d \neq 0$ $\forall a, p, q \in \mathbb{Z}$ where p is of the form $4k+3$ and q is of the form $4l+1$.

Let $p = 4k+3$ and $q = 4l+1$. Then we have,

$$\begin{aligned} r &= pq + p + q \\ &= (4k+3)(4l+1) + (4k+3) + (4l+1) \\ &= 16kl + 4k + 12l + 3 + 4k + 3 + 4l + 1 \\ &= 16kl + 8k + 16l + 7 \\ &= 4(4kl + 2k + 4l + 1) + 3 \\ &= 4m + 3. \end{aligned}$$

$m = 4kl + 2k + 4l + 1$. So we see r is of the form $4h+3$. Using Theorem 1, for some $x, y \in \mathbb{Z}$ where $y \neq 0$,

$$\begin{aligned} \sigma^{\star}(2^a r) - 2^a r &= \sigma^{\star}(2^a)\sigma^{\star}(r) - 2^a r \\ &= (x + yi)(r+1) - 2^a r \\ &= (x(r+1) - 2^a r) + y(r+1)i \\ &= c + di. \end{aligned}$$

Since where $c = x(r+1) - 2^a r$ and $d = y(r+1)$. Since $y \neq 0$, then $d \neq 0$ so $c + di$ has a non-zero imaginary part as desired.

The proof of Case 4, where p is of the form $4k+1$ and q is of the form $4l+3$ is similar to Case 3. Since it was shown in each of the cases outlined above that $\sigma^\star(2^a r) - 2^a r$ will always result in a Gaussian integer under the specified conditions, then $\sigma^\star(m) - m$ will always result in a Gaussian integer when we consider $m = 2^a r$ and will never equal the integer $n = 2^a pq$ in $\mathbb{Z}$. So there are no $(2,1)$ pairs of the form $(2^n pq, 2^n r)$ in $\mathbb{Z}$ that are also amicable in $\mathbb{Z}_i$. $\square$

Similar proofs could be constructed for pairs of different types to illustrate that there are many pairs in the integers that will not carry over to the Gaussian integers. Doing so would be tedious and non exhaustive for all existing pairs. So we now ask the question, are there any pairs in the integers that will *always* carry over to the Gaussian integers? This question is explored in the following section.

Pairs that Will Always Carry Over

The previous sections illustrated the importance of understanding the limitations on pairs carrying over from the integers to Gaussian integers based on their different factorizations in $\mathbb{Z}_i$. If a pair contains a power of 2^n in $\mathbb{Z}$, it will factor as $(1+i)^{2n}\epsilon$ in $\mathbb{Z}_i$. If a pair contains an odd prime p of the form $4k+1$ then such a p will not be prime in $\mathbb{Z}_i$ and can be written as the product of Gaussian integers where $p = (a+bi)(b+ai)\epsilon$ and $p = N(a+bi)$. In these situations the factorizations of these integers will cause the complex sum of divisors function $\sigma^\star$ to give a different output than the regular sum of divisors function. Recall, however, that if an odd prime p is of the form $4k+3$ then p will also be prime as a Gaussian integer. This concept leads to the following theorem.

Theorem 3. *Let (m, n) be amicable in $\mathbb{Z}$. If $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_s^{\alpha_s}$ and $n = q_1^{\beta_1} \cdot q_2^{\beta_2} \cdot \ldots \cdot q_t^{\beta_t}$ where all of the p_i and q_j are of the form $4k+3$, then (m, n) is amicable in $\mathbb{Z}_i$.*

Proof. Consider $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_s^{\alpha_s}$ and $n = q_1^{\beta_1} \cdot q_2^{\beta_2} \cdot \ldots \cdot q_t^{\beta_t}$. Since each p_i is of the form $4k+3$ the prime factorization of m in the Gaussian integers is the same as its factorization in the integers. But (m, n) is amicable in $\mathbb{Z}$, so:

$$\begin{aligned} \sigma^\star(m) - m &= \sigma(m) - m \\ &= n \end{aligned}$$

and

$$\begin{aligned} \sigma^\star(n) - n &= \sigma(n) - n \\ &= m \end{aligned}$$

Hence (m, n) is also amicable in $\mathbb{Z}_i$. $\square$

The smallest pair satisfying the criteria for Theorem 3 is the pair $(294706414233, 305961592167)$ which was discovered by TeRiele in 1995 [4]. In this case, $294706414233 = 3^4 \cdot 7^2 \cdot 11 \cdot 19 \cdot 47 \cdot 7559$ and $305961592167 = 3^4 \cdot 7 \cdot 11 \cdot 19 \cdot 971 \cdot 2659$. Each prime in the factorization of each number is of the form $4k+3$, hence the prime factorization for both of these numbers is the same in the Gaussian integers as it is in the integers. Now applying the complex sum of divisors function we get:

$$\begin{aligned}
\sigma^{\star}(294706414233) &= \sigma(294706414233) \\
&= \sigma(3^4)\sigma(7^2)\sigma(11)\sigma(19)\sigma(47)\sigma(7559) \\
&= \frac{3^5-1}{3-1} \cdot \frac{7^3-1}{7-1} \cdot 12 \cdot 20 \cdot 48 \cdot 7560 \\
&= 121 \cdot 57 \cdot 12 \cdot 20 \cdot 48 \cdot 7560 \\
&= 600668006400
\end{aligned}$$

Then $600668006400 - 294706414233 = 305961592167$. Similarly:

$$\begin{aligned}
\sigma^{\star}(305961592167) &= \sigma(305961592167) \\
&= \sigma(3^4)\sigma(7)\sigma(11)\sigma(19)\sigma(971)\sigma(2659) \\
&= \frac{3^5-1}{3-1} \cdot 8 \cdot 12 \cdot 20 \cdot 972 \cdot 2660 \\
&= 121 \cdot 8 \cdot 12 \cdot 20 \cdot 972 \cdot 2660 \\
&= 600668006400
\end{aligned}$$

Then $600668006400 - 305961592167 = 294706414233$. So we see that $(294706414233, 305961592167)$ is also amicable in the Gaussian integers.

Theorem 3 and the previous example illustrate pairs in the integers that will always carry over to the Gaussian integers. From this point on we place our focus on amicable pairs that exist within the Gaussian integers without respect to a counterpart pair in $\mathbb{Z}$. We seek to answer the question: Are there Gaussian amicable pairs?

Part 4

Gaussian Amicable Pairs

A Formula for 2^n

We begin our search for Gaussian amicable pairs by searching for formulas for powers of small primes using the complex sum of divisors function $\sigma^\star$. We start with the smallest prime 2. The following *Table 1* shows values for $\sigma^\star(2^n)$ for a small number of n values.

TABLE 1

Values of $\sigma^\star(2^n)$

n	$\sigma^\star(2^n)$
1	2+3i
2	-4+5i
3	-8-7i
4	16-15i
5	32+33i
6	-64+65i
7	-128-127i
8	256-255i
9	512+513i
10	-1024+1025i

If we look at the sequences formed by the whole numbers and the imaginary parts then we see that they can be obtained from the second diagonal of Pascal's triangle with binomial coefficient exponents on -1 of the form $\binom{k}{2}$. By adjusting the starting points of the binomial coefficients we obtain the following theorem.

Theorem 1. *Let $\sigma^\star$ denote the complex sum of divisors function. Let n be an integer greater than or equal to 1. Then*

$$\sigma^\star(2^n) = (-1)^{\binom{n+4}{2}} 2^n + (-1)^{\binom{n+3}{2}} (2^n + (-1)^{\binom{n+3}{2}})i$$

Proof: Take the base case $n = 1$. Then,

$$\begin{aligned}
\sigma^\star(2^1) &= \sigma^\star\big((1+i)^2\big) \\
&= \frac{(1+i)^{2+1} - 1}{(1+i) - 1} \\
&= \frac{(1+i)^3 - 1}{i} \\
&= \frac{-2 + 2i - 1}{i} \\
&= \frac{-3 + 2i}{i} \\
&= 2 + 3i
\end{aligned}$$

and

$$\begin{aligned}
(-1)^{\binom{1+4}{2}} 2^1 + (-1)^{\binom{1+3}{2}} (2^1 + (-1)^{\binom{1+3}{2}})i &= (-1)^{\binom{5}{2}} 2 + (-1)^{\binom{4}{2}} (2 + (-1)^{\binom{4}{2}})i \\
&= \big((-1)^{10} \cdot 2\big) + \big((-1)^6 (2 + (-1)^6)\big)i \\
&= (2 \cdot 1) + (1 \cdot (2 + 1))i \\
&= (2) + (1 \cdot 3)i \\
&= 2 + 3i
\end{aligned}$$

So the formula holds true for the base case $n = 1$.

Now assume the statement holds true for $n = k$.That is,

$$\sigma^\star(2^k) = (-1)^{\binom{k+4}{2}} 2^k + (-1)^{\binom{k+3}{2}} (2^k + (-1)^{\binom{k+3}{2}}) i \tag{1}$$

First we manipulate $\sigma^\star(2^n)$ as follows:

$$\begin{aligned}
\sigma^\star(2^n) &= \sigma^\star\big((1+i)^{2n}\big) \\
&= \frac{(1+i)^{2n+1} - 1}{i} \\
&= \frac{(1+i)^{2n}(1+i) - 1}{i}
\end{aligned}$$

Then if we multiply both sides by i we have,

$$\begin{aligned}
i\sigma^\star(2^n) &= (1+i)^{2n}(1+i) - 1 \\
&= (1+i)^{2n+1} - 1
\end{aligned}$$

This implies that

$$i\sigma^\star(2^n) + 1 = (1+i)^{2n+1} \tag{2}$$

Now we consider, the statement $\sigma^\star(2^{k+1})$ for $n = k + 1$.

$$\begin{aligned}
\sigma^\star(2^{k+1}) &= \frac{(1+i)^{2(k+1)+1} - 1}{(1+i) - 1} \\
&= \frac{(1+i)^{2k+3} - 1}{i} \\
&= \frac{(1+i)^{2k+1}(1+i)^2 - 1}{i} \\
&= \frac{\big(i\sigma^\star(2^k) + 1\big)(1+i)^2 - 1}{i} && \text{by substituting (2)}
\end{aligned}$$

Then,

$$\begin{aligned}
\sigma^\star(2^{k+1}) &= \frac{\big(i\sigma^\star(2^k)+1\big)(1+i)^2}{i} - \frac{1}{i} \\
&= \frac{\big(i\sigma^\star(2^k)+1\big)(2i)}{i} + i \\
&= \big(i\sigma^\star(2^k)+1\big)(2) + i \\
&= \big(2+2\sigma^\star(2^k)i\big) + i \\
&= 2 + \big(2\sigma^\star(2^k)+1\big)i
\end{aligned}$$

Now that this expression is simplified into a Gaussian integer, we can use our induction hypothesis and we have:

$$\sigma^\star(2^{k+1}) = 2 + \left(2\left[(-1)^{\binom{k+4}{2}}2^k + (-1)^{\binom{k+3}{2}}\big(2^k + (-1)^{\binom{k+3}{2}}\big)i\right] + 1\right)i$$

Simplifying we see,

$$\begin{aligned}
\sigma^\star(2^{k+1}) &= 2 + \left(2\left[(-1)^{\binom{k+4}{2}}2^k + (-1)^{\binom{k+3}{2}}\big(2^k + (-1)^{\binom{k+3}{2}}\big)i\right] + 1\right)i \\
&= 2 + \left(2^{k+1}(-1)^{\binom{k+4}{2}} + \left(2(-1)^{\binom{k+3}{2}}\big[2^k + (-1)^{\binom{k+3}{2}}\big]\right)i + 1\right)i \\
&= 2 + \left(\left[2^{k+1}(-1)^{\binom{k+4}{2}} + 1\right] + \left[2(-1)^{\binom{k+3}{2}}\big(2^k + (-1)^{\binom{k+3}{2}}\big)\right]i\right)i \\
&= 2 + \left(2(-1)(-1)^{\binom{k+3}{2}}\big[2^k + (-1)^{\binom{k+3}{2}}\big]\right) + \left(2^{k+1}(-1)^{\binom{k+4}{2}} + 1\right)i \\
&= \left(2(-1)(-1)^{\binom{k+3}{2}}\big[2^k + (-1)^{\binom{k+3}{2}}\big] + 2\right) + \left(2^{k+1}(-1)^{\binom{k+4}{2}} + 1\right)i \\
&= \left(2^{k+1}(-1)(-1)^{\binom{k+3}{2}} + 2(-1)(-1)^{\binom{k+3}{2}}(-1)^{\binom{k+3}{2}} + 2\right) \\
&\qquad\qquad + \left(2^{k+1}(-1)^{\binom{k+4}{2}} + 1\right)i \\
&= \left(2^{k+1}(-1)(-1)^{\binom{k+3}{2}} + 2(-1)(-1)^{2\binom{k+3}{2}} + 2\right) \\
&\qquad\qquad + \left(2^{k+1}(-1)^{\binom{k+4}{2}} + 1\right)i
\end{aligned}$$

Now we expand the binomial exponents,

$$\begin{aligned}
\sigma^{\star}(2^{k+1}) &= 2\left(2^k(-1)(-1)^{\frac{(k+3)(k+2)}{2}} + (-1)(-1)^{2(\frac{(k+3)(k+2)}{2})} + 1\right)\\
&\qquad + \left(2^{k+1}(-1)^{\frac{(k+4)(k+3)}{2}} + 1\right)i\\
&= 2\left(2^k(-1)(-1)^{\frac{(k+3)(k+2)}{2}} + (-1)(-1)^{2(\frac{(k+3)(k+2)}{2})} + (-1)^{2(\frac{(k+3)(k+2)}{2})}\right)\\
&\qquad + \left(2^{k+1}(-1)^{\frac{(k+4)(k+3)}{2}} + (-1)^{2(\frac{(k+4)(k+3)}{2})}\right)i
\end{aligned}$$

Because -1 raised to any even power is still 1, we can manipulate the 1 term in the real and imaginary part above.

Now simplifying we see,

$$\begin{aligned}
\sigma^{\star}(2^{k+1}) &= 2\left(2^k(-1)(-1)^{\frac{(k+3)(k+2)}{2}} + (-1)^{2(\frac{(k+3)(k+2)}{2})}\big[(-1) + 1\big]\right)\\
&\qquad + \left(2^{k+1}(-1)^{\frac{(k+4)(k+3)}{2}} + (-1)^{\frac{(k+4)(k+3)}{2}}(-1)^{\frac{(k+4)(k+3)}{2}}\right)i\\
&= 2\left(2^k(-1)(-1)^{\frac{(k+3)(k+2)}{2}} + (-1)^{2(\frac{(k+3)(k+2)}{2})}(0)\right)\\
&\qquad + \left((-1)^{\frac{(k+4)(k+3)}{2}}\left[2^{k+1} + (-1)^{\frac{(k+4)(k+3)}{2}}\right]\right)i\\
&= 2\left(2^k(-1)(-1)^{\frac{(k+3)(k+2)}{2}}\right)\\
&\qquad + \left((-1)^{\frac{(k+4)(k+3)}{2}}\left[2^{k+1} + (-1)^{\frac{(k+4)(k+3)}{2}}\right]\right)i\\
&= \left(2^{k+1}(-1)(-1)^{\frac{(k+3)(k+2)}{2}}\right)\\
&\qquad + \left((-1)^{\binom{k+4}{2}}\left[2^{k+1} + (-1)^{\binom{k+4}{2}}\right]\right)i
\end{aligned}$$

Note that the imaginary part is equivalent to the imaginary part for the proposed formula for $\sigma^{\star}(2^{k+1})$. It remains that we must now finish simplifying the real part.

$$\sigma^\star(2^{k+1}) = \left(2^{k+1}(-1)(-1)^{\frac{(k+3)(k+2)}{2}}(1)\right) + \left((-1)^{\binom{k+4}{2}}\left[2^{k+1} + (-1)^{\binom{k+4}{2}}\right]\right)i$$
$$= \left(2^{k+1}(-1)(-1)^{\frac{(k+3)(k+2)}{2}}(-1)^{2(\frac{2k+6}{2})}\right) + \left((-1)^{\binom{k+4}{2}}\left[2^{k+1} + (-1)^{\binom{k+4}{2}}\right]\right)i$$

Once again, we can manipulate the constant 1 in the real part above because $(-1)^{2(\frac{2k+6}{2})} = 1$ since we are raising -1 to an even power. So,

$$\sigma^\star(2^{k+1}) = \left(2^{k+1}(-1)^{\left(1+\frac{(k+3)(k+2)}{2}+2(\frac{2k+6}{2})\right)}\right) + \left((-1)^{\binom{k+4}{2}}\left[2^{k+1} + (-1)^{\binom{k+4}{2}}\right]\right)i$$
$$= \left(2^{k+1}(-1)^{\left(\frac{2}{2}+\frac{(k+3)(k+2)}{2}+\frac{4k+12}{2}\right)}\right) + \left((-1)^{\binom{k+4}{2}}\left[2^{k+1} + (-1)^{\binom{k+4}{2}}\right]\right)i$$
$$= \left(2^{k+1}(-1)^{\left(\frac{2+(k+3)(k+2)+4k+12)}{2}\right)}\right) + \left((-1)^{\binom{k+4}{2}}\left[2^{k+1} + (-1)^{\binom{k+4}{2}}\right]\right)i$$
$$= \left(2^{k+1}(-1)^{\left(\frac{k^2+9k+20}{2}\right)}\right) + \left((-1)^{\binom{k+4}{2}}\left[2^{k+1} + (-1)^{\binom{k+4}{2}}\right]\right)i$$
$$= \left(2^{k+1}(-1)^{\left(\frac{(k+5)(k+4)}{2}\right)}\right) + \left((-1)^{\binom{k+4}{2}}\left[2^{k+1} + (-1)^{\binom{k+4}{2}}\right]\right)i$$
$$= \left(2^{k+1}(-1)^{\binom{k+5}{2}}\right) + \left((-1)^{\binom{k+4}{2}}\left[2^{k+1} + (-1)^{\binom{k+4}{2}}\right]\right)i$$

So we see by induction that the formula holds for $n = k + 1$. Hence, by mathematical induction, it follows that,

$$\sigma^\star(2^n) = (-1)^{\binom{n+4}{2}}2^n + (-1)^{\binom{n+3}{2}}(2^n + (-1)^{\binom{n+3}{2}})i \quad \text{for all } n \quad \square$$

What is the significance of finding a formula for $\sigma^\star(2^n)$ and other small primes? Because amicable pairs are a numerical property linked to the sum of divisors function, and because they are often classified according to their common factors, and from there on their type, it is necessary to be able to manipulate the sum of divisors function. In this case, if we know how the complex sum of divisors function will behave, we can use this to our advantage by inputting combinations of certain powers of primes when we search for pairs. When dealing with Gaussian integers this is certainly more efficient, for we could begin with powers of primes in the integers and already have an idea of what their output would be from the $\sigma^\star$ function which would most likely have an imaginary part. Hence, a formula for $\sigma^\star(2^n)$ is a good start.

However, finding formulas for powers of five, thirteen, seventeen, etc. proved to be extremely messy and there were no logical patterns. Due to the lack of consistency in the outputs of the complex sum of divisors function $\sigma^\star$ for powers of these small primes, it became necessary to pursue another option for finding amicable pairs in $\mathbb{Z}_i$. Instead of a specific computer search involving inputing different values for different powers of small primes into formulas, a general computer search was done. This is discussed in detail in the following section.

Computer Search for Pairs in $\mathbb{Z}_i$

In order to search for amicable pairs in $\mathbb{Z}_i$ it is almost necessary to use the computer as an aid. Even in the strategy outlined in the previous section, the reasoning for finding specific formulas for powers of small primes centered around being able to program computers to find pairs more efficiently. In this section we discuss methods for a general computer search for pairs, some of which focus on finding pairs with specific common factors.

Suppose we want to search for pairs with a common factor of $(1+i)^8$. We can write a short program in Mathematica that lets $x = (1+i)^8 \cdot (a+bi)$ where a is incremented from 1 to $1,000,000$ and b is incremented from 1 to $100,000$. Then we can let $y = \sigma^\star(x) - x$. Similarly we can also define a variable z where $z = \sigma^\star(y) - y$. Then if $z = x$ the result will be an amicable pair. Similar programs can be written to find pairs with different common factors by replacing the $(1+i)^8$ with the appropriate factors. Below is an example of the code used in Mathematica® to find amicable pairs with a common factor of $(1+i)^8$.

```
For [a = 1, a < 1000000, a++, Print ["a = ", a ];
For [b = 1, b < 100000, b++, x = (1 + i)^8 · (a + bi);
y = DivisorSigma [1, x, GaussianIntegers → True] −x;
z = DivisorSigma [1, y, GaussianIntegers → True] −y;
If [z == x, Print [x, " and " , y," are amicable",
"where the first number has a factor of (1 + i)^8]]]]
```

A similar search can be done to perform an even more general search for Gaussian amicable pairs. In this case instead of having a common factor attached to the $a + bi$ in our code, we simply start with $x = a + bi$ and increment a from 1 to $1,000,000$ and b from 1 to $100,000$. If given enough computers, it is possible to extend this range even further. In these cases we would still define the variables y and z as they are above. These methods led to the discovery of over one hundred new Gaussian amicable pairs! *Table 2* below briefly summarizes the pairs found arranged by common factors.

TABLE 2

Pairs Organized by Common Factor

Common Factor	Number Found
$(1+i)^7$	22
$(1+i)^8$	12
$(1+i)^9$	4
$(1+i)^m(1+2i)^n$	4
$(1+2i)$	14
$(1+2i)^2$	15
$(1+2i)^3$	11
$(1+2i)^4$	1
$(1+2i)^m(1+4i)^n$	14
Total	97

After classifying the pairs above according to their common factors it was straightforward to then organize the pairs according to type. We found a wide range of different types of pairs, ranging from $(2,1)$ to $(5,5)$. Exotic pairs were found as well. For example,

$$\begin{cases} 736 - 16560i = (1+i)^8(45+2i)(23)(-i) \\ 17648 + 768i = (1+i)^8(1103+48i) \end{cases}$$

is an example of a $(2,1)$ pair found,

$$\begin{cases} 2335041 + 13975712i = (1+2i)^3(4+5i)(7+2i)(5+6i)(1+16i)(150+157i) \\ 15760959 - 1495712i = (1+2i)^3(1+4i)(3+8i)(14+15i)(26+i)(38+65i)(-i) \end{cases}$$

is an example of the only $(5,5)$ pair found, and

$$\begin{cases} 301559 - 146012i = (1+2i)^3(1+4i)(71+16i)(57+82i) \\ -142839 - 241828i = (1+2i)(1+4i)(7+8i)(13+22i)(16+111i)(-i) \end{cases}$$

is an example of an exotic pair that was found after the computer search. An exhaustive list of all pairs found arranged by type can be found in Appendix B. *Table 3* below indicates the types of pairs found and their frequency amongst the results.

TABLE 3

All Pairs Found

Type	Number Found
(2,1)	3
(2,2)	19
(3,2)	42
(3,3)	14
(4,2)	4
(4,3)	7
(4,4)	5
(5,3)	4
(5,5)	1
exotic	12
Total	111

Part 5

Results

Amicable Pairs in the Integers that Carry Over

The first question that we sought to answer was whether or not there were amicable pairs in the integers that were also amicable in the Gaussian Integers. A preliminary search returned a quick answer—not many. It was shown specifically that the smallest pair $(220, 284)$ was not amicable in $\mathbb{Z}_i$. This was because the different factorizations of both numbers in the pair caused the complex sum of divisors function to yield a Gaussian integer with an imaginary part in both cases. Using this concept a proof was provided that in fact there are no pairs of the form $(2^n pq, 2^n r)$ in $\mathbb{Z}$ that are also amicable in $\mathbb{Z}_i$.

A proof similar to this one could be provided to show that there are certain $(2, 2)$ pairs, $(3, 2)$ pairs, $(3, 3)$ pairs, etc. in the integers that do not carry over in the Gaussian integers, the primary reasoning being that when you apply the complex sum of divisors $\sigma^\star$ function to the first member of the pair you will get an Gaussian integer $a + bi$ where $b \neq 0$. Hence it will be impossible for $\sigma^\star(m) - m = n$ in such a case for a pair (m, n) in the integers. Trying to construct such a proof for all amicable pairs in the integers would be tedious and non exhaustive as new pairs are always being found. Therefore it was necessary to instead focus on whether or not there existed pairs in the integers that would *always* carry over to the integers.

Any odd prime integer p that is of the form $4k + 3$ will also be prime in the Gaussian integers. Therefore if two members of an amicable pair in $\mathbb{Z}$ have prime factorizations consisting entirely of primes of this form then these pairs must also be amicable in $\mathbb{Z}$. Their factorizations will not change and hence the $\sigma^\star$ function will give the same output as the regular σ function. This proof was provided in earlier sections along with examples. Hence there are pairs in the integers that will *always* always carry over to the Gaussian integers.

If there are other pairs in the integers that carry over they must satisfy certain criteria. Such a pair, for instance, must consist of prime factors that contain some combination of primes of the $4k+1$, $4k+3$, and/or powers of 2 so that when the $\sigma^{\star}$ function is applied the result is of the form $a+bi$ where $b=0$. If this happens for both members of the pair then it will be possible for such a pair to also be amicable in $\mathbb{Z}_i$. However, up to this point, such pairs have not been found.

More Gaussian Amicable Pairs

A general computer search for Gaussian amicable pairs combined with a specific search for pairs with certain common factors resulted in over one hundred amicable pairs in $\mathbb{Z}_i$ being discovered. These pairs were then categorized according to their types and common factors. Though an impressive number of pairs were found, there still remain a great deal more to be discovered.

For example, when using the computer to find pairs with certain common factors, we focused on small powers of $(1+i)^l, (1+2i)^m$, and $(1+4i)^n$. These are representative of powers of $2, 5$, and 17 in the integers. While we found many pairs programming the computer this way, this search was by no means comprehensive. We could have just as easily searched for pairs with common factors of $(1+6i)^j, (4+5i)^k, \ldots$ and so on as long as the prime p in $\mathbb{Z}_i$ has a norm of the form $4k+1$. Also, just as there was a search done for pairs with a combination of different powers of factors, like powers of $(1+2i)$ and $(1+4i)$, we could do similar searches on new factors, or perhaps extend the search to pairs that have three, four, five, $\ldots$ different factors in common. The possibilities are endless. This paper provides a basis for the methodology that can be used for future searches for more pairs.

It is also important to note methods for finding amicable pairs of certain types in $\mathbb{Z}_i$. When just looking for pairs with certain common factors we can get results of all different types, $(2,1), (2,2), (3,2)$, etc. Is there a way to generate pairs of a specific type? In the computer search outlined in previous sections, the output was always a set of two Gaussian integers $a+bi$ and $c+di$ that were amicable. However, searching for pairs in such a way will never guarantee that they are of a specific type. Though they may have had a common factor, upon factoring the $a+bi$ and $c+di$ we often found other common factors that were not detailed in the initial code. This affects the "types" of Gaussian amicable pairs found.

It was mentioned in the introduction that over time many mathematicians, such as Euler and Thābit ibn Qurra, have developed methods for finding $(2,1)$ pairs. Modern mathematicians have done the same. Adapting these methods for Gaussian amicable pairs is possible, but programming the computer is not as "simple" as when just dealing with integers. Dealing with units and factoring Gaussian integers makes programming a somewhat straightforward idea extremely complicated. This could become less tedious if a program was first adapted from the factoring algorithm mentioned in previous sections. However, because this type of programming was not the primary focus of the research done in this paper, we leave adaptations of such methods to future study for both this author and the reader.

Natural Extensions

A natural extension for research on Gaussian amicable pairs is to investigate whether or not there exist Gaussian aliquot sequences. An aliquot sequence is one where the sum of divisors function of the first number subtracted by that number gives the second number of the sequence. Then the sum of divisors functions of that number subtracted by the second number gives the third number of the sequence, and so on. That is, if we let $s(n) = \sigma(n) - n$, then the sequence $s^0(n) = n, s^1(n) = s(n), s^2(n) = s(s(n)), \ldots$ is called an aliquot sequence. There are several options for how an aliquot sequence will terminate.

If a given sequence ends at zero, then it considered to be bounded. Similarly, if a sequence becomes periodic it is considered bounded as well. For example if a sequence reaches a period of two then those two numbers are an amicable pair. If a sequence reaches a period of three or more then those numbers are called *sociable*. If, however, a sequence just reaches a constant then this constant is called a perfect number. If a sequence reaches a constant but that constant is not perfect it is referred to as an *aspiring number*. Such a sequence is still bounded because it becomes periodic with a period of one. There are also several aliquot sequences where it has not yet been determined if the sequence terminates.

An aliquot sequence in the Gaussian integers can be identified by using $s^\star(n) = \sigma^\star(n) - n$. Then the sequence $s_0^\star(n) = n, s_1^\star(n) = s^\star(n), s_2^\star(n) = s^\star(s^\star(n)), \ldots$. As we have shown there exist Gaussian amicable pairs, we know that there are already aliquot sequences of period two in the Gaussian integers. What remains to be investigated is whether or not there exist sociable and/or aspiring Gaussian integers. Similarly, the question can be posed as to whether or not there are any aliquot sequences in the integers that are also aliquot sequences in the Gaussian integers. These are interesting tasks that we leave to the reader.

Bibliography

[1] Erica Flapan and Tim Marks and James Pommersheim. Number Theory: A Lively Introduction with Proofs, Applications, and Stories, John Wiley & Sons, Hoboken, NJ, 2010 pp.599-649

[2] Gupta. S. "Amicable Numbers." From http://www.shyamsundergupta.com/amicable.htm

[3] Koshy, Thomas. 2^{nd} edition., Elementary Number Theory with Applications. Academic Press, 2007.

[4] Pedersen, Jan. (Last update 28-Sep-2007). "Known Amicable Pairs." From http://amicable.homepage.dk/knwnc2.htm

[5] Spira, Robert, "The Complex Sum of Divisors", The American Mathematical Monthly, v. 68, February 1961, pp. 120-124

[6] Weisstein, Eric W. "Aliquot Sequence." From *Math World*–A Wolfram Web Resource. http://mathworld.wolfram.com/AliquotSequence.html

[7] Weisstein, Eric W. "Amicable Pair." From *Math World*–A Wolfram Web Resource. http://mathworld.wolfram.com/AmicablePair.html

[8] Weisstein. E. "Euler's Rule." From *Math World*–A Wolfram Web Resource. http://mathworld.wofram.com/EulersRule.html

Appendices

Appendix A

Pairs Organized by Type

(2,1) Pairs

$$\begin{cases} -21246 - 8807i = (1+2i)(1+4i)(6+11i)(2+3i)(45+32i)(-i) \\ 5166 - 26953i = (1+2i)(1+4i)(6+11i)(41+234i) \end{cases}$$

$$\begin{cases} 736 - 16560i = (1+i)^8(45+2i)(23)(-i) \\ 17648 + 768i = (1+i)^8(1103+48i) \end{cases}$$

$$\begin{cases} -1036624 + 495520i = (1+i)^8(2+27i)(28+25i)(63+32i) \\ 536656 + 1058336i = (1+i)^8(2+27i)(1055+2528i)(-i) \end{cases}$$

(2,2) Pairs

$$\begin{cases} -6468-5251i=(1+2i)^2(1+4i)(1+14i)(27+10i)(-i) \\ 5356-6133i=(1+2i)^2(1+4i)(3+8i)(36+29i) \end{cases}$$

$$\begin{cases} 42696+4120i=(1+i)^7(13+2i)(215+192i) \\ 4104-43720i=(1+i)^7(20+3i)(143+128i)(-i) \end{cases}$$

$$\begin{cases} 50632-14568i=(1+i)^7(11+4i)(393+62i) \\ -15832-52632i=(1+i)^7(25+12i)(175+8i)(-i) \end{cases}$$

$$\begin{cases} 272776+159240i=(1+i)^7(8+7i)(2175+1472i) \\ 154424-298440i=(1+i)^7(79+120i)(199+56i)(-i) \end{cases}$$

$$\begin{cases} 287864-25560i=(1+i)^7(11+6i)(1999+400i) \\ -35864-298440i=(1+i)^7(34+5i)(671+384i)(-i) \end{cases}$$

$$\begin{cases} 170176+125296i=(1+i)^8(35+48i)(67+212i)(-i) \\ -130384+164960i=(1+i)^8(5+18i)(415+568i) \end{cases}$$

$$\begin{cases} 64000+15248i=(1+i)^8(17+168i)(23+8i)(-i) \\ -15952+65248i=(1+i)^8(23+48i)(61+50i) \end{cases}$$

$$\begin{cases} 53312-10800i=(1+i)^8(61+76i)(31+16i)(-i) \\ 10336+54064i=(1+i)^8(10+29i)(111+16i) \end{cases}$$

$$\begin{cases} 292528+103008i=(1+i)^8(95+24i)(197+18i) \\ 101888-277968i=(1+i)^8(15+4i)(1187+108i)(-i) \end{cases}$$

$$\begin{cases} 636256 + 48656i = (1+i)^8(17+2i)(95+2328i)(-i) \\ -51664 + 668800i = (1+i)^8(85+128i)(215+168i) \end{cases}$$

$$\begin{cases} 10336 + 54064i = (1+i)^8(10+29i)(111+16i) \\ 53312 - 10800i = (1+i)^8(31+16i)(61+76i)(-i) \end{cases}$$

$$\begin{cases} 397888 + 544688i = (1+i)^8(3+52i)(439+680i)(-i) \\ -551488 + 368912i = (1+i)^8(4+15i)(863+2528i) \end{cases}$$

$$\begin{cases} 260368 + 1316848i = (1+i)^9(11+20i)(1195+2308i)(-i) \\ -1329840 + 303056i = (1+i)^9(63+92i)(215+496i) \end{cases}$$

$$\begin{cases} 288528 + 701168i = (1+i)^9(11+20i)(915+1148i)(-i) \\ -703600 + 313936i = (1+i)^9(73+162i)(135+136i) \end{cases}$$

$$\begin{cases} 896400 + 696944i = (1+i)^9(11+20i)(2043+812i)(-i) \\ -681808 + 931312i = (1+i)^9(43+102i)(447+112i) \end{cases}$$

$$\begin{cases} 636256 + 48656i = (1+i)^8(17+2i)(95+2328i)(-i) \\ -51664 + 668800i = (1+i)^8(85+128i)(215+168i) \end{cases}$$

$$\begin{cases} 16072 + 14712i = (1+i)^7(11+4i)(63+152i) \\ 15128 - 17112i = (1+i)^7(4+65i)(31)(-i) \end{cases}$$

$$\begin{cases} 8008 + 3960i = (1+i)^7(23+68i)(11) \\ 4232 - 8280i = (1+i)^7(11+34i)(23)(-i) \end{cases}$$

$$\begin{cases} 69760 - 16432i = (1+i)^8(13+8i)(207+208i)(-i) \\ 18848 + 70928i = (1+i)^8(38+143i)(31) \end{cases}$$

(3,2) Pairs

$$\begin{cases} -259222 - 59439i = (1+2i)(1+4i)(2+3i)(4+11i)(653+202i)(-i) \\ 12022 - 309201i = (1+2i)(1+4i)(4+9i)(1343+3132i) \end{cases}$$

$$\begin{cases} -3235 + 1020i = (1+2i)(2+i)^3(5+4i)(7+20i)(-i) \\ -3549 - 4988i = (1+2i)(1+10i)(15+272i) \end{cases}$$

$$\begin{cases} -4694 + 467i = (1+2i)^2(4+i)(4+5i)(11+34i)(-i) \\ -766 - 6187i = (1+2i)^2(1+6i)(116+169i) \end{cases}$$

$$\begin{cases} -14612 - 7159i = (1+2i)^2(6+i)(1+4i)(7+8i)(10+7i)(-i) \\ 4212 - 19241i = (1+2i)^2(6+i)(1+14i)(23+40i) \end{cases}$$

$$\begin{cases} -1895 + 2060i = (1+2i)^2(2+i)(13+8i)(13+10i)(-i) \\ -3433 - 2356i = (1+2i)^2(7+12i)(53+28i) \end{cases}$$

$$\begin{cases} -3970 + 2435i = (1+2i)^2(2+i)(8+3i)(21+44i)(-i) \\ -4478 - 5471i = (1+2i)^2(8+13i)(65+66i) \end{cases}$$

$$\begin{cases} -24877 - 15664i = (1+2i)^2(1+4i)(2+3i)(289+270i)(-i) \\ 9877 - 27536i = (1+2i)^2(3+2i)^3(59+110i) \end{cases}$$

$$\begin{cases} -24766 - 12687i = (1+2i)^2(6+i)(1+4i)(5+4i)(24+25i)(-i) \\ 7766 - 33313i = (1+2i)^2(6+i)(1+14i)(39+70i) \end{cases}$$

$$\begin{cases} 168373 - 417664i = (1+2i)^2(4+5i)(7+10i)(23+12i)(38+23i) \\ -365173 - 221936i = (1+2i)^2(4+5i)(1+4i)(2203+2372i)(-i) \end{cases}$$

$$\begin{cases} -203672 - 30529i = (1+2i)^2(6+i)(3+2i)(1+4i)(384+245i)(-i) \\ -48328 - 270471i = (1+2i)^2(6+i)(41+70i)(49+100i) \end{cases}$$

$$\begin{cases} -657187 - 231884i = (1+2i)^2(6+i)(3+2i)(1+4i)(1119+1060i)(-i) \\ 14387 - 928516i = (1+2i)^2(6+i)(22+43i)(199+600i) \end{cases}$$

$$\begin{cases} -503402 + 18861i = (1+2i)^2(6+i)(3+2i)(1+4i)(1034+415i)(-i) \\ -237398 - 628261i = (1+2i)^2(6+i)(21+46i)(295+322i) \end{cases}$$

$$\begin{cases} 293124 - 106057i = (1+2i)^3(15+2i)(1+4i)(8+13i)(29+4i) \\ -108484 - 289463i = (1+2i)^3(15+2i)(2+3i)^2(123+68i)(-i) \end{cases}$$

$$\begin{cases} -42529 - 11098i = (1+2i)(1+4i)(2+3i)(5+24i)(53+10i)(-i) \\ 5953 - 51974i = (1+2i)(1+4i)(1+10i)(341+450i) \end{cases}$$

$$\begin{cases} 82599 - 121532i = (1+2i)^2(1+4i)(10+7i)(15+2i)(20+33i) \\ -121359 - 98788i = (1+2i)^2(1+4i)(2+5i)(1109+870i)(-i) \end{cases}$$

$$\begin{cases} -22233 - 7876i = (1+2i)(1+4i)(2+3i)(5+8i)(61+44i)(-i) \\ 3033 - 28124i = (1+2i)(1+4i)(9+10i)(5+228i) \end{cases}$$

$$\begin{cases} -259222 - 59439i = (1+2i)(1+4i)(2+3i)(4+11i)(653+202i)(-i) \\ 12022 - 309201i = (1+2i)(1+4i)(4+9i)(1343+3132i) \end{cases}$$

$$\begin{cases} -5479 - 396528i = (1+2i)^2(1+4i)(8+13i)(33+2i)(38+3i) \\ -392801 - 15432i = (1+2i)^2(1+4i)(4+5i)(2831+924i)(-i) \end{cases}$$

$$\begin{cases} 274360 + 248216i = (1+i)^7(7+12i)(110+57i)(19) \\ 262040 - 309416i = (1+i)^7(3+32i)(1115+12i)(-i) \end{cases}$$

$$\begin{cases} 246392 + 198968i = (1+i)^7(13+10i)(62+65i)(19) \\ 215608 - 268568i = (1+i)^7(10+19i)(1319+520i)(-i) \end{cases}$$

$$\begin{cases} 306360 + 250424i = (1+i)^7(13+22i)(54+25i)(23) \\ 255240 - 329624i = (1+i)^7(31+16i)(599+870i)(-i) \end{cases}$$

$$\begin{cases} 65480 + 166952i = (1+i)^7(23+8i)(4+25i)(25+6i) \\ 176920 - 78152i = (1+i)^7(13+58i)(239+160i)(-i) \end{cases}$$

$$\begin{cases} 102120 + 136712i = (1+i)^7(4+21i)(29+10i)(23) \\ 142680 - 115112i = (1+i)^7(23+48i)(259+160i)(-i) \end{cases}$$

$$\begin{cases} 50104 + 212856 = (1+i)^7(17+2i)(14+25i)(23+32i) \\ 228296 - 61656i = (1+i)^7(15+38i)(319+400i)(-i) \end{cases}$$

$$\begin{cases} 157320 + 240616i = (1+i)^7(3+20i)(62+23i)(19) \\ 248280 - 178216i = (1+i)^7(59+40i)(159+344i)(-i) \end{cases}$$

$$\begin{cases} 151080 + 289816i = (1+i)^7(17+2i)(5+18i)(81+40i) \\ 305400 - 176776i = (1+i)^7(11+56i)(491+240i)(-i) \end{cases}$$

$$\begin{cases} 8136 + 424120i = (1+i)^7(13+8i)(25+14i)(25+82i) \\ 459384 - 25480i = (1+i)^7(11+40i)(531+824i)(-i) \end{cases}$$

$$\begin{cases} 168568 + 324600i = (1+i)^7(24+5i)(25+4i)(3+52i) \\ 336632 - 183000i = (1+i)^7(26+11i)(135+1192i)(-i) \end{cases}$$

$$\begin{cases} 146312 + 362792i = (1+i)^7(15+4i)(9+26i)(72+37i) \\ 384088 - 171032i = (1+i)^7(23+40i)(508+625i)(-i) \end{cases}$$

$$\begin{cases} 130504+418904i=(1+i)^7(17+10i)(13+28i)(59+24i) \\ 439016-161864i=(1+i)^7(90+77i)(95+336i)(-i) \end{cases}$$

$$\begin{cases} 132968+435240i=(1+i)^7(11+4i)(2+45i)(75+14i) \\ 466072-155160i=(1+i)^7(16+29i)(743+1080i)(-i) \end{cases}$$

$$\begin{cases} 304072+391848i=(1+i)^7(7+18i)(23+8i)(92+15i) \\ 398648-347208i=(1+i)^7(11+204i)(227+28i)(-i) \end{cases}$$

$$\begin{cases} -36792+756968i=(1+i)^7(9+10i)(51+4i)(8+97i) \\ 810792-6968i=(1+i)^7(39+200i)(199+290i)(-i) \end{cases}$$

$$\begin{cases} -438464+375280i=(1+i)^8(20+7i)(3+40i)(35+24i) \\ 390416+467744i=(1+i)^8(2+15i)(1343+2128i)(-i) \end{cases}$$

$$\begin{cases} -1118256+70544i=(1+i)^9(2+3i)(3+88i)(35+152i)(-i) \\ 67184+120560i=(1+i)^9(4+21i)(255+128i)(-i) \end{cases}$$

$$\begin{cases} -369976-109432i=(1+i)^7(1+2i)^2(1+4i)(5+2i)(11+6i)(5+24i)(-i) \\ -461624-95768i=(1+i)^7(1+2i)^2(1+4i)(11+14i)(35+108i)(-i) \end{cases}$$

$$\begin{cases} -215223-417336i=(1+2i)^3(4+5i)(6+19i)(104+35i)(3)(-i) \\ 551223-310664i=(1+2i)^3(4+5i)(83+28i)(34+95i) \end{cases}$$

$$\begin{cases} -442001-234332i=(1+2i)^2(1+4i)(24+35i)(63+52i)(7)(-i) \\ 261201-516068i=(1+2i)^2(1+4i)(79+40i)(123+292i) \end{cases}$$

$$\begin{cases} 181032+274360i=(1+i)^7(3+20i)(71+26i)(19) \\ 282168-204760i=(1+i)^7(63+32i)(134+415i)(-i) \end{cases}$$

$$\begin{cases} 246392 + 198968i = (1+i)^7(13+10i)(62+65i)(19) \\ 215608 - 268568i = (1+i)^7(10+19i)(1319+520i)(-i) \end{cases}$$

$$\begin{cases} 306360 + 250424i = (1+i)^7(13+22i)(54+25i)(23) \\ 255240 - 329624i = (1+i)^7(31+16i)(599+870i)(-i) \end{cases}$$

$$\begin{cases} 20868 + 34476i = (1+i)^5(1+2i)^3(1+14i)(15+2i)(3)(-i) \\ 20732 + 48724i = (1+i)^5(1+2i)^3(1+4i)(183+88i)(-i) \end{cases}$$

(3,3) Pairs

$$\begin{cases} -629476 - 461397i = (1+2i)(1+6i)(1+4i)(4+i)^2(431+696i)(-i) \\ 439396 - 963243i = (1+2i)(1+6i)(2+15i)(31+44i)(93+22i) \end{cases}$$

$$\begin{cases} -349539 - 233248i = (1+2i)^2(6+i)(1+6i)(1+4i)(541+104i)(-i) \\ 158339 - 410352i = (1+2i)^2(6+i)(3+8i)(7+18i)(84+25i) \end{cases}$$

$$\begin{cases} 57942 - 44431i = (1+2i)^3(1+4i)(24+25i)(45+8i) \\ -41942 - 47569i = (1+2i)^3(5+2i)(4+5i)(94+135i)(-i) \end{cases}$$

$$\begin{cases} 265256 - 173533i = (1+2i)^3(5+2i)(1+4i)(13+12i)(5+72i)(-i) \\ 205144 + 250333i = (1+2i)^3(5+2i)(5+6i)(1+24i)(14+25i) \end{cases}$$

$$\begin{cases} 1053996 - 84353i = (1+2i)^3(5+2i)(1+4i)(17+42i)(1+94i)(-i) \\ 140244 + 1052033i = (1+2i)^3(5+2i)(2+5i)(7+38i)(11+84i) \end{cases}$$

$$\begin{cases} -185562 - 570259i = (1+2i)^2(1+4i)(11+14i)(13+28i)(49+20i)(-i) \\ 576442 - 179581i = (1+2i)^2(1+4i)(1+14i)(20+7i)(81+56i) \end{cases}$$

$$\begin{cases} -381134 - 431113i = (1+2i)^2(1+4i)(80+11i)(19+20i)(6+11i)(-i) \\ 426334 - 387687i = (1+2i)^2(1+4i)(7+12i)(31+26i)(48+13i) \end{cases}$$

$$\begin{cases} 687846 - 97153i = (1+2i)^3(1+4i)(4+5i)(35+8i)(61+24i) \\ -143846 - 718847i = (1+2i)^3(1+4i)(4+15i)(20+i)(51+4i)(-i) \end{cases}$$

$$\begin{cases} -583631 - 441812i = (1+2i)(1+4i)(2+3i)(1+6i)(3107+1858i)(-i) \\ 283631 - 750028i = (1+2i)(1+4i)(12+7i)(14+9i)(11+376i) \end{cases}$$

$$\begin{cases} 716246 + 602097i = (1+2i)^2(1+4i)(3+8i)(2+65i)(63+52i) \\ 578954 - 766097i = (1+2i)^2(1+4i)(1+24i)(31+26i)(19+44i)(-i) \end{cases}$$

$$\begin{cases} 582448 + 34161i = (1+2i)^3(5+2i)(2+7i)(5+22i)(59) \\ 70352 - 552561i = (1+2i)^3(5+2i)(1+4i)(29+6i)(29+70i)(-i) \end{cases}$$

$$\begin{cases} 237888 + 399616i = (1+2i)^3(1+i)^{12}(1+6i)(8+13i)(7)(-i) \\ 181312 + 422784i = (1+2i)^3(1+i)^{12}(1+4i)(4+5i)(23+8i)(-i) \end{cases}$$

$$\begin{cases} 44764 + 38148i = (1+i)^5(1+2i)^2(1+6i)(23+18i)(11+4i)(-i) \\ 46436 + 36252i = (1+i)^5(1+2i)^2(5+6i)(1+14i)(19)(-i) \end{cases}$$

$$\begin{cases} -157567 - 36594i = (1+2i)^2(2+5i)(1+4i)(75+16i)(19)(-i) \\ 12127 - 162286i = (1+2i)^2(2+5i)(15+2i)(10+13i)(23+8i) \end{cases}$$

(4,2) Pairs

$$\begin{cases} -779326 - 277267i = (1+2i)(1+4i)(2+3i)(10+3i)(29+10i)(41+66i)(-i) \\ 131326 - 1090733i = (1+2i)(1+4i)(8+45i)(1399+2200i) \end{cases}$$

$$\begin{cases} -1105 + 1020i = (1+2i)(2+i)(4+i)(1+4i)(12+13i)(-i) \\ -2639 - 1228i = (1+2i)(5+22i)(25+52i) \end{cases}$$

$$\begin{cases} -294413 - 125726i = (1+2i)(1+4i)(2+3i)(13+2i)(13+22i)(25+14i)(-i) \\ 69773 - 413794i = (1+2i)(1+4i)(23+32i)(83+1152i) \end{cases}$$

$$\begin{cases} -6880 + 4275i = (1+2i)(2+i)(3+2i)^2(2+7i)(17+2i)(-i) \\ -13056 - 9187i = (1+2i)(6+19i)(67+352i) \end{cases}$$

(4,3) Pairs

$$\begin{cases} -12449 - 2978i = (1+2i)(1+4i)(3+2i)(2+5i)(53+48i)(-i) \\ -991 - 16702i = (1+2i)(10+13i)(2+5i)(11+84i) \end{cases}$$

$$\begin{cases} 104319 - 140692i = (1+2i)^3(8+13i)(29+6i)(24+25i) \\ -92319 - 75308i = (1+2i)^3(4+i)(6+i)(8+3i)(13+48i)(-i) \end{cases}$$

$$\begin{cases} -498059 - 299648i = (1+2i)(1+4i)(2+3i)(1+14i)(17+2i)(71+16i)(-i) \\ 221579 - 675712i = (1+2i)(1+4i)(15+32i)(13+38i)(53+12i) \end{cases}$$

$$\begin{cases} -61678 - 28721i = (1+2i)^2(3+2i)(1+4i)(14+i)(38+53i)(-i) \\ 14518 - 81999i = (1+2i)^2(5+2i)(4+29i)(69+80i) \end{cases}$$

$$\begin{cases} -537556 - 455477i = (1+2i)(3+10i)(2+3i)(1+4i)(38+23i)(45+8i)(-i) \\ 368116 - 754603i = (1+2i)(3+10i)(4+5i)(1+14i)(371+150i) \end{cases}$$

$$\begin{cases} -8547 + 4606i = (1+2i)(2+3i)(1+4i)(29+30i)(7)(-i) \\ -8733 - 10366i = (1+2i)(1+6i)(1+14i)(71) \end{cases}$$

$$\begin{cases} -95718 + 18549i = (1+2i)^2(1+4i)(18+5i)(64+55i)(3)(-i) \\ -47962 - 130309i = (1+2i)^2(1+14i)(16+i)(103+68i) \end{cases}$$

(4,4) Pairs

$$\begin{cases} -339374 - 103253i = (1+2i)(2+3i)(1+4i)(1+6i)(1724+325i)(-i) \\ 3374 - 388747i = (1+2i)(4+5i)(5+8i)(3+10i)(251+114i) \end{cases}$$

$$\begin{cases} 471431 - 6558i = (1+2i)^3(7+2i)(9+10i)(1+14i)(29+10i) \\ 40569 - 409442i = (1+2i)^3(3+2i)(4+5i)(11+20i)(61+34i)(-i) \end{cases}$$

$$\begin{cases} 307084 - 217837i = (1+2i)^3(1+4i)(7+2i)(1+14i)(55+58i)(-i) \\ 220916 + 281837i = (1+2i)^3(5+2i)(1+6i)(8+7i)(19+90i) \end{cases}$$

$$\begin{cases} 129276 - 57893i = (1+2i)^3(6+i)(5+6i)(1+14i)(19) \\ -33276 - 134107i = (1+2i)^3(1+4i)(5+2i)(5+8i)(59)(-i) \end{cases}$$

(5,3) Pairs

$$\begin{cases} -31606 - 9787i = (1+2i)(2+3i)(1+4i)(6+5i)(10+3i)(10+7i)(-i) \\ -6074 - 47573i = (1+2i)(5+4i)(1+26i)(38+123i) \end{cases}$$

$$\begin{cases} -30638 - 951i = (1+2i)(2+3i)(1+4i)(5+2i)(10+3i)(13+10i)(-i) \\ -18322 - 44169i = (1+2i)(7+8i)(14+15i)(3+98i) \end{cases}$$

$$\begin{cases} -251422 + 39081i = (1+2i)(3+2i)(1+4i)(4+5i)(13+8i)(78+7i)(-i) \\ -168578 - 331081i = (1+2i)(2+5i)(29+110i)(219+160i) \end{cases}$$

$$\begin{cases} -211300 + 18485i = (1+2i)(2+i)(1+4i)(7+2i)(22+25i)(35+24i)(-i) \\ -149084 - 342773i = (1+2i)(1+10i)(95+24i)(16+169i) \end{cases}$$

(5,5) Pairs

$$\begin{cases} 2335041 + 13975712i = (1+2i)^3(4+5i)(7+2i)(5+6i)(1+16i)(150+157i) \\ 15760959 - 1495712i = (1+2i)^3(1+4i)(3+8i)(14+15i)(26+i)(38+65i)(-i) \end{cases}$$

Exotic Pairs

$$\begin{cases} 13484 - 10787i = (1+2i)^3(5+4i)(3+8i)(26+11i) \\ -9004 - 12573i = (1+2i)(1+4i)(5+8i)(1+10i)(13+12i)(-i) \end{cases}$$

$$\begin{cases} 29281 - 19358i = (1+2i)^3(3+2i)(14+15i)(24+35i) \\ -13681 - 29842i = (1+2i)^2(1+6i)(4+5i)(11+6i)(9+10i)(-i) \end{cases}$$

$$\begin{cases} 9912 - 45641i = (1+2i)^3(1+4i)(5+4i)(61+146i)(-i) \\ 43528 + 9161i = (1+2i)(1+4i)(1+6i)(1+10i)(30+73i) \end{cases}$$

$$\begin{cases} 61908 - 42619i = (1+2i)^3(1+6i)(11+14i)(62+3i) \\ -37428 - 47141i = (1+2i)(2+5i)(8+3i)(1+10i)(5+58i)(-i) \end{cases}$$

$$\begin{cases} 361049 - 154582i = (1+2i)^3(5+2i)(2+7i)(568+693i) \\ -109529 - 330378i = (1+2i)^2(5+2i)(2+7i)(3+8i)(135+158i)(-i) \end{cases}$$

$$\begin{cases} 301559 - 146012i = (1+2i)^3(1+4i)(71+16i)(57+82i) \\ -142839 - 241828i = (1+2i)(1+4i)(7+8i)(13+22i)(16+111i)(-i) \end{cases}$$

$$\begin{cases} 95568 - 430624i = (1+i)^8(1+2i)^3(2+3i)(1+6i)(79+80i)(-i) \\ 50352 - 398816i = (1+i)^8(1+2i)(2+3i)^3(1+6i)(23+32i)(-i) \end{cases}$$

$$\begin{cases} 391696 - 737328i = (1+i)^9(1+2i)^3(2+5i)(1+4i)(147+22i)(-i) \\ 258064 - 702992i = (1+i)^9(1+2i)(2+5i)(1+6i)(3+10i)(28+33i)(-i) \end{cases}$$

$$\begin{cases} 10028 + 23596i = (1+i)^5(1+2i)^3(2+3i)(79+80i)(-i) \\ 11092 + 20564i = (1+i)^5(1+2i)(2+3i)^2(133+50i) \end{cases}$$

$$\begin{cases} -452528 - 233821i = (1+2i)^2(3+2i)(1+4i)(59+44i)(85+38i)(-i) \\ 155248 - 563619i = (1+2i)(1+4i)(9+14i)(13+22i)(130+73i) \end{cases}$$

$$\begin{cases} -721368 - 250336i = (1+i)^6(1+2i)(1+4i)(1+10i)(1+16i)(23+60i)(-i) \\ -626472 - 223904i = (1+i)^6(1+2i)^3(4+i)(8+3i)(3+10i)(3+20i)(-i) \end{cases}$$

$$\begin{cases} -119184 + 45912i = (1+i)^6(1+2i)^2(1+4i)(1+6i)(24+35i)(3)(-i) \\ -153616 + 110888i = (1+i)^6(1+2i)^3(4+5i)(43+328i)(-i) \end{cases}$$

Appendix B

Pairs Organized by Common Factors

Common Factor of $(1+i)^7$

$$\begin{cases} 8008+3960i=(1+i)^7(23+68i)(11) \\ 4232-8280i=(1+i)^7(11+34i)(23)(-i) \end{cases}$$

$$\begin{cases} 274360+248216i=(1+i)^7(7+12i)(110+57i)(19) \\ 262040-309416i=(1+i)^7(3+32i)(1115+12i)(-i) \end{cases}$$

$$\begin{cases} 42696+4120i=(1+i)^7(13+2i)(215+192i) \\ 4104-43720i=(1+i)^7(20+3i)(143+128i)(-i) \end{cases}$$

$$\begin{cases} 246392+198968i=(1+i)^7(13+10i)(62+65i)(19) \\ 215608-268568i=(1+i)^7(10+19i)(1319+520i)(-i) \end{cases}$$

$$\begin{cases} 306360+250424i=(1+i)^7(13+22i)(54+25i)(23) \\ 255240-329624i=(1+i)^7(31+16i)(599+870i)(-i) \end{cases}$$

$$\begin{cases} 50632-14568i=(1+i)^7(11+4i)(393+62i) \\ -15832-52632i=(1+i)^7(25+12i)(175+8i)(-i) \end{cases}$$

$$\begin{cases} 272776+159240i=(1+i)^7(8+7i)(2175+1472i) \\ 154424-298440i=(1+i)^7(79+120i)(199+56i)(-i) \end{cases}$$

$$\begin{cases} 176920-78152i=(1+i)^7(13+58i)(239+160i)(-i) \\ 65480+166952i=(1+i)^7(23+8i)(4+25i)(25+6i) \end{cases}$$

$$\begin{cases} 142680-115112i=(1+i)^7(23+48i)(259+160i)(-i) \\ 102120+136712i=(1+i)^7(4+21i)(29+10i)(23) \end{cases}$$

$$\begin{cases} 228296 - 61656i = (1+i)^7(15+38i)(319+400i)(-i) \\ 50104 + 212856 = (1+i)^7(17+2i)(14+25i)(23+32i) \end{cases}$$

$$\begin{cases} 287864 - 25560i = (1+i)^7(11+6i)(1999+400i) \\ -35864 - 298440i = (1+i)^7(34+5i)(671+384i)(-i) \end{cases}$$

$$\begin{cases} 248280 - 178216i = (1+i)^7(59+40i)(159+344i)(-i) \\ 157320 + 240616i = (1+i)^7(3+20i)(62+23i)(19) \end{cases}$$

$$\begin{cases} 305400 - 176776i = (1+i)^7(11+56i)(491+240i)(-i) \\ 151080 + 289816i = (1+i)^7(17+2i)(5+18i)(81+40i) \end{cases}$$

$$\begin{cases} 459384 - 25480i = (1+i)^7(11+40i)(531+824i)(-i) \\ 8136 + 424120i = (1+i)^7(13+8i)(25+14i)(25+82i) \end{cases}$$

$$\begin{cases} 282168 - 204760i = (1+i)^7(63+32i)(134+415i)(-i) \\ 181032 + 274360i = (1+i)^7(3+20i)(71+26i)(19) \end{cases}$$

$$\begin{cases} 336632 - 183000i = (1+i)^7(26+11i)(135+1192i)(-i) \\ 168568 + 324600i = (1+i)^7(24+5i)(25+4i)(3+52i) \end{cases}$$

$$\begin{cases} 384088 - 171032i = (1+i)^7(23+40i)(508+625i)(-i) \\ 146312 + 362792i = (1+i)^7(15+4i)(9+26i)(72+37i) \end{cases}$$

$$\begin{cases} 439016 - 161864i = (1+i)^7(90+77i)(95+336i)(-i) \\ 130504 + 418904i = (1+i)^7(17+10i)(13+28i)(59+24i) \end{cases}$$

$$\begin{cases} 466072 - 155160i = (1+i)^7(16+29i)(743+1080i)(-i) \\ 132968 + 435240i = (1+i)^7(11+4i)(2+45i)(75+14i) \end{cases}$$

$$\begin{cases} 398648 - 347208i = (1+i)^7(11+204i)(227+28i)(-i) \\ 304072 + 391848i = (1+i)^7(7+18i)(23+8i)(92+15i) \end{cases}$$

$$\begin{cases} 810792 - 6968i = (1+i)^7(39+200i)(199+290i)(-i) \\ -36792 + 756968i = (1+i)^7(9+10i)(51+4i)(8+97i) \end{cases}$$

$$\begin{cases} 16072 + 14712i = (1+i)^7(11+4i)(63+152i) \\ 15128 - 17112i = (1+i)^7(4+65i)(31)(-i) \end{cases}$$

Common Factor of $(1+i)^8$

$$\begin{cases} 17648 + 768i = (1+i)^8(1103+48i) \\ 736 - 16560i = (1+i)^8(45+2i)(23)(-i) \end{cases}$$

$$\begin{cases} 170176 + 125296i = (1+i)^8(35+48i)(67+212i)(-i) \\ -130384 + 164960i = (1+i)^8(5+18i)(415+568i) \end{cases}$$

$$\begin{cases} 64000 + 15248i = (1+i)^8(17+168i)(23+8i)(-i) \\ -15952 + 65248i = (1+i)^8(23+48i)(61+50i) \end{cases}$$

$$\begin{cases} 53312 - 10800i = (1+i)^8(61+76i)(31+16i)(-i) \\ 10336 + 54064i = (1+i)^8(10+29i)(111+16i) \end{cases}$$

$$\begin{cases} 69760 - 16432i = (1+i)^8(13+8i)(207+208i)(-i) \\ 18848 + 70928i = (1+i)^8(38+143i)(31) \end{cases}$$

$$\begin{cases} 292528 + 103008i = (1+i)^8(95+24i)(197+18i) \\ 101888 - 277968i = (1+i)^8(15+4i)(1187+108i)(-i) \end{cases}$$

$$\begin{cases} 636256 + 48656i = (1+i)^8(17+2i)(95+2328i)(-i) \\ -51664 + 668800i = (1+i)^8(85+128i)(215+168i) \end{cases}$$

$$\begin{cases} 10336 + 54064i = (1+i)^8(10+29i)(111+16i) \\ 53312 - 10800i = (1+i)^8(31+16i)(61+76i)(-i) \end{cases}$$

$$\begin{cases} 390416 + 467744i = (1+i)^8(2+15i)(1343+2128i)(-i) \\ -438464 + 375280i = (1+i)^8(20+7i)(3+40i)(35+24i) \end{cases}$$

$$\begin{cases} 397888 + 544688i = (1+i)^8(3+52i)(439+680i)(-i) \\ -551488 + 368912i = (1+i)^8(4+15i)(863+2528i) \end{cases}$$

$$\begin{cases} 536656 + 1058336i = (1+i)^8(2+27i)(1055+2528i)(-i) \\ -1036624 + 495520i = (1+i)^8(2+27i)(28+25i)(63+32i) \end{cases}$$

$$\begin{cases} 636256 + 48656i = (1+i)^8(17+2i)(95+2328i)(-i) \\ -51664 + 668800i = (1+i)^8(85+128i)(215+168i) \end{cases}$$

Common Factor of $(1+i)^9$

$$\begin{cases} 896400 + 696944i = (1+i)^9(11+20i)(2043+812i)(-i) \\ -681808 + 931312i = (1+i)^9(43+102i)(447+112i) \end{cases}$$

$$\begin{cases} 67184 + 120560i = (1+i)^9(4+21i)(255+128i)(-i) \\ -1118256 + 70544i = (1+i)^9(2+3i)(3+88i)(35+152i)(-i) \end{cases}$$

$$\begin{cases} 260368 + 1316848i = (1+i)^9(11+20i)(1195+2308i)(-i) \\ -1329840 + 303056i = (1+i)^9(63+92i)(215+496i) \end{cases}$$

$$\begin{cases} 288528 + 701168i = (1+i)^9(11+20i)(915+1148i)(-i) \\ -703600 + 313936i = (1+i)^9(73+162i)(135+136i) \end{cases}$$

Common Factor of $(1+i)^m(1+2i)^n$

$$\begin{cases} 237888+399616i=(1+2i)^3(1+i)^{12}(1+6i)(8+13i)(7)(-i) \\ 181312+422784i=(1+2i)^3(1+i)^{12}(1+4i)(4+5i)(23+8i)(-i) \end{cases}$$

$$\begin{cases} 20868+34476i=(1+i)^5(1+2i)^3(1+14i)(15+2i)(3)(-i) \\ 20732+48724i=(1+i)^5(1+2i)^3(1+4i)(183+88i)(-i) \end{cases}$$

$$\begin{cases} 44764+38148i=(1+i)^5(1+2i)^2(1+6i)(23+18i)(11+4i)(-i) \\ 46436+36252i=(1+i)^5(1+2i)^2(5+6i)(1+14i)(19)(-i) \end{cases}$$

$$\begin{cases} -119184+45912i=(1+i)^6(1+2i)^2(1+4i)(1+6i)(24+35i)(3)(-i) \\ -153616+110888i=(1+i)^6(1+2i)^3(4+5i)(43+328i)(-i) \end{cases}$$

$$\begin{cases} -461624-95768i=(1+i)^7(1+2i)^2(1+4i)(11+14i)(35+108i)(-i) \\ -369976-109432i=(1+i)^7(1+2i)^2(1+4i)(5+2i)(11+6i)(5+24i)(-i) \end{cases}$$

Common Factor of $1 + 2i$

$$\begin{cases} -3235 + 1020i = (1 + 2i)(2 + i)^3(5 + 4i)(7 + 20i)(-i) \\ -3549 - 4988i = (1 + 2i)(1 + 10i)(15 + 272i) \end{cases}$$

$$\begin{cases} -1105 + 1020i = (1 + 2i)(2 + i)(4 + i)(1 + 4i)(12 + 13i)(-i) \\ -2639 - 1228i = (1 + 2i)(5 + 22i)(25 + 52i) \end{cases}$$

$$\begin{cases} -31606 - 9787i = (1 + 2i)(2 + 3i)(1 + 4i)(6 + 5i)(10 + 3i)(10 + 7i)(-i) \\ -6074 - 47573i = (1 + 2i)(5 + 4i)(1 + 26i)(38 + 123i) \end{cases}$$

$$\begin{cases} -30638 - 951i = (1 + 2i)(2 + 3i)(1 + 4i)(5 + 2i)(10 + 3i)(13 + 10i)(-i) \\ -18322 - 44169i = (1 + 2i)(7 + 8i)(14 + 15i)(3 + 98i) \end{cases}$$

$$\begin{cases} -12449 - 2978i = (1 + 2i)(1 + 4i)(3 + 2i)(2 + 5i)(53 + 48i)(-i) \\ -991 - 16702i = (1 + 2i)(10 + 13i)(2 + 5i)(11 + 84i) \end{cases}$$

$$\begin{cases} -8547 + 4606i = (1 + 2i)(2 + 3i)(1 + 4i)(29 + 30i)(7)(-i) \\ -8733 - 10366i = (1 + 2i)(1 + 6i)(1 + 14i)(71) \end{cases}$$

$$\begin{cases} -21246 - 8807i = (1 + 2i)(1 + 4i)(6 + 11i)(2 + 3i)(45 + 32i)(-i) \\ 5166 - 26953i = (1 + 2i)(1 + 4i)(6 + 11i)(41 + 234i) \end{cases}$$

$$\begin{cases} -211300 + 18485i = (1 + 2i)(2 + i)(1 + 4i)(7 + 2i)(22 + 25i)(35 + 24i)(-i) \\ -149084 - 342773i = (1 + 2i)(1 + 10i)(95 + 24i)(16 + 169i) \end{cases}$$

$$\begin{cases} 12022 - 309201i = (1 + 2i)(1 + 4i)(4 + 9i)(1343 + 3132i) \\ -259222 - 59439i = (1 + 2i)(1 + 4i)(2 + 3i)(4 + 11i)(653 + 202i)(-i) \end{cases}$$

$$\begin{cases} -629476 - 461397i = (1+2i)(1+6i)(1+4i)(4+i)^2(431+696i)(-i) \\ 439396 - 963243i = (1+2i)(1+6i)(2+15i)(31+44i)(93+22i) \end{cases}$$

$$\begin{cases} -339374 - 103253i = (1+2i)(2+3i)(1+4i)(1+6i)(1724+325i)(-i) \\ 3374 - 388747i = (1+2i)(4+5i)(5+8i)(3+10i)(251+114i) \end{cases}$$

$$\begin{cases} -537556 - 455477i = (1+2i)(3+10i)(2+3i)(1+4i)(38+23i)(45+8i)(-i) \\ 368116 - 754603i = (1+2i)(3+10i)(4+5i)(1+14i)(371+150i) \end{cases}$$

$$\begin{cases} -251422 + 39081i = (1+2i)(3+2i)(1+4i)(4+5i)(13+8i)(78+7i)(-i) \\ -168578 - 331081i = (1+2i)(2+5i)(29+110i)(219+160i) \end{cases}$$

$$\begin{cases} -6880 + 4275i = (1+2i)(2+i)(3+2i)^2(2+7i)(17+2i)(-i) \\ -13056 - 9187i = (1+2i)(6+19i)(67+352i) \end{cases}$$

Common Factor of $(1+2i)^2$

$$\begin{cases} -4694+467i=(1+2i)^2(4+i)(4+5i)(11+34i)(-i) \\ -766-6187i=(1+2i)^2(1+6i)(116+169i) \end{cases}$$

$$\begin{cases} -14612-7159i=(1+2i)^2(6+i)(1+4i)(7+8i)(10+7i)(-i) \\ 4212-19241i=(1+2i)^2(6+i)(1+14i)(23+40i) \end{cases}$$

$$\begin{cases} -1895+2060i=(1+2i)^2(2+i)(13+8i)(13+10i)(-i) \\ -3433-2356i=(1+2i)^2(7+12i)(53+28i) \end{cases}$$

$$\begin{cases} -3970+2435i=(1+2i)^2(2+i)(8+3i)(21+44i)(-i) \\ -4478-5471i=(1+2i)^2(8+13i)(65+66i) \end{cases}$$

$$\begin{cases} -61678-28721i=(1+2i)^2(3+2i)(1+4i)(14+i)(38+53i)(-i) \\ 14518-81999i=(1+2i)^2(5+2i)(4+29i)(69+80i) \end{cases}$$

$$\begin{cases} -24877-15664i=(1+2i)^2(1+4i)(2+3i)(289+270i)(-i) \\ 9877-27536i=(1+2i)^2(3+2i)^3(59+110i) \end{cases}$$

$$\begin{cases} -24766-12687i=(1+2i)^2(6+i)(1+4i)(5+4i)(24+25i)(-i) \\ 7766-33313i=(1+2i)^2(6+i)(1+14i)(39+70i) \end{cases}$$

$$\begin{cases} -95718+18549i=(1+2i)^2(1+4i)(18+5i)(64+55i)(3)(-i) \\ -47962-130309i=(1+2i)^2(1+14i)(16+i)(103+68i) \end{cases}$$

$$\begin{cases} -157567-36594i=(1+2i)^2(2+5i)(1+4i)(75+16i)(19)(-i) \\ 12127-162286i=(1+2i)^2(2+5i)(15+2i)(10+13i)(23+8i) \end{cases}$$

$$\begin{cases} -349539 - 233248i = (1+2i)^2(6+i)(1+6i)(1+4i)(541+104i)(-i) \\ 158339 - 410352i = (1+2i)^2(6+i)(3+8i)(7+18i)(84+25i) \end{cases}$$

$$\begin{cases} -365173 - 221936i = (1+2i)^2(4+5i)(1+4i)(2203+2372i)(-i) \\ 168373 - 417664i = (1+2i)^2(4+5i)(7+10i)(23+12i)(38+23i) \end{cases}$$

$$\begin{cases} -203672 - 30529i = (1+2i)^2(6+i)(3+2i)(1+4i)(384+245i)(-i) \\ -48328 - 270471i = (1+2i)^2(6+i)(41+70i)(49+100i) \end{cases}$$

$$\begin{cases} -294413 - 125726i = (1+2i)(1+4i)(2+3i)(13+2i)(13+22i)(25+14i)(-i) \\ 69773 - 413794i = (1+2i)(1+4i)(23+32i)(83+1152i) \end{cases}$$

$$\begin{cases} -657187 - 231884i = (1+2i)^2(6+i)(3+2i)(1+4i)(1119+1060i)(-i) \\ 14387 - 928516i = (1+2i)^2(6+i)(22+43i)(199+600i) \end{cases}$$

$$\begin{cases} -503402 + 18861i = (1+2i)^2(6+i)(3+2i)(1+4i)(1034+415i)(-i) \\ -237398 - 628261i = (1+2i)^2(6+i)(21+46i)(295+322i) \end{cases}$$

Common Factor of $(1+2i)^3$

$$\begin{cases} 57942-44431i=(1+2i)^3(1+4i)(24+25i)(45+8i) \\ -41942-47569i=(1+2i)^3(5+2i)(4+5i)(94+135i)(-i) \end{cases}$$

$$\begin{cases} 582448+34161i=(1+2i)^3(5+2i)(2+7i)(5+22i)(59) \\ 70352-552561i=(1+2i)^3(5+2i)(1+4i)(29+6i)(29+70i)(-i) \end{cases}$$

$$\begin{cases} 129276-57893i=(1+2i)^3(6+i)(5+6i)(1+14i)(19) \\ -33276-134107i=(1+2i)^3(1+4i)(5+2i)(5+8i)(59)(-i) \end{cases}$$

$$\begin{cases} 471431-6558i=(1+2i)^3(7+2i)(9+10i)(1+14i)(29+10i) \\ 40569-409442i=(1+2i)^3(3+2i)(4+5i)(11+20i)(61+34i)(-i) \end{cases}$$

$$\begin{cases} 293124-106057i=(1+2i)^3(15+2i)(1+4i)(8+13i)(29+4i) \\ -108484-289463i=(1+2i)^3(15+2i)(2+3i)^2(123+68i)(-i) \end{cases}$$

$$\begin{cases} 104319-140692i=(1+2i)^3(8+13i)(29+6i)(24+25i) \\ -92319-75308i=(1+2i)^3(4+i)(6+i)(8+3i)(13+48i)(-i) \end{cases}$$

$$\begin{cases} 265256-173533i=(1+2i)^3(5+2i)(1+4i)(13+12i)(5+72i)(-i) \\ 205144+250333i=(1+2i)^3(5+2i)(5+6i)(1+24i)(14+25i) \end{cases}$$

$$\begin{cases} 307084-217837i=(1+2i)^3(1+4i)(7+2i)(1+14i)(55+58i)(-i) \\ 220916+281837i=(1+2i)^3(5+2i)(1+6i)(8+7i)(19+90i) \end{cases}$$

$$\begin{cases} 1053996-84353i=(1+2i)^3(5+2i)(1+4i)(17+42i)(1+94i)(-i) \\ 140244+1052033i=(1+2i)^3(5+2i)(2+5i)(7+38i)(11+84i) \end{cases}$$

$$\begin{cases} 551223 - 310664i = (1+2i)^3(4+5i)(83+28i)(34+95i) \\ -215223 - 417336i = (1+2i)^3(4+5i)(6+19i)(104+35i)(3)(-i) \end{cases}$$

$$\begin{cases} 2335041 + 13975712i = (1+2i)^3(4+5i)(7+2i)(5+6i)(1+16i)(150+157i) \\ 15760959 - 1495712i = (1+2i)^3(1+4i)(3+8i)(14+15i)(26+i)(38+65i)(-i) \end{cases}$$

Common Factor of $(1+2i)^4$

$$\begin{cases} 946241 - 292888i = (1+2i)^4(2+3i)(9+10i)(15+22i)(29+10i)(-i) \\ 300559 + 980488i = (1+2i)^4(1+4i)(8+3i)(27+20i)(24+25i) \end{cases}$$

Common Factor of $(1+2i)^n(1+4i)^n$

$$\begin{cases} -42529-11098i=(1+2i)(1+4i)(2+3i)(5+24i)(53+10i)(-i) \\ 5953-51974i=(1+2i)(1+4i)(1+10i)(341+450i) \end{cases}$$

$$\begin{cases} -6468-5251i=(1+2i)^2(1+4i)(1+14i)(27+10i)(-i) \\ 5356-6133i=(1+2i)^2(1+4i)(3+8i)(36+29i) \end{cases}$$

$$\begin{cases} -185562-570259i=(1+2i)^2(1+4i)(11+14i)(13+28i)(49+20i)(-i) \\ 576442-179581i=(1+2i)^2(1+4i)(1+14i)(20+7i)(81+56i) \end{cases}$$

$$\begin{cases} -381134-431113i=(1+2i)^2(1+4i)(80+11i)(19+20i)(6+11i)(-i) \\ 426334-387687i=(1+2i)^2(1+4i)(7+12i)(31+26i)(48+13i) \end{cases}$$

$$\begin{cases} -121359-98788i=(1+2i)^2(1+4i)(2+5i)(1109+870i)(-i) \\ 82599-121532i=(1+2i)^2(1+4i)(10+7i)(15+2i)(20+33i) \end{cases}$$

$$\begin{cases} 687846-97153i=(1+2i)^3(1+4i)(4+5i)(35+8i)(61+24i) \\ -143846-718847i=(1+2i)^3(1+4i)(4+15i)(20+i)(51+4i)(-i) \end{cases}$$

$$\begin{cases} -22233-7876i=(1+2i)(1+4i)(2+3i)(5+8i)(61+44i)(-i) \\ 3033-28124i=(1+2i)(1+4i)(9+10i)(5+228i) \end{cases}$$

$$\begin{cases} -583631-441812i=(1+2i)(1+4i)(2+3i)(1+6i)(3107+1858i)(-i) \\ 283631-750028i=(1+2i)(1+4i)(12+7i)(14+9i)(11+376i) \end{cases}$$

$$\begin{cases} -259222-59439i=(1+2i)(1+4i)(2+3i)(4+11i)(653+202i)(-i) \\ 12022-309201i=(1+2i)(1+4i)(4+9i)(1343+3132i) \end{cases}$$

$$\begin{cases} -442001 - 234332i = (1+2i)^2(1+4i)(24+35i)(63+52i)(7)(-i) \\ 261201 - 516068i = (1+2i)^2(1+4i)(79+40i)(123+292i) \end{cases}$$

$$\begin{cases} 716246 + 602097i = (1+2i)^2(1+4i)(3+8i)(2+65i)(63+52i) \\ 578954 - 766097 = (1+2i)^2(1+4i)(1+24i)(31+26i)(19+44i)(-i) \end{cases}$$

$$\begin{cases} -498059 - 299648i = (1+2i)(1+4i)(2+3i)(1+14i)(17+2i)(71+16i)(-i) \\ 221579 - 675712i = (1+2i)(1+4i)(15+32i)(13+38i)(53+12i) \end{cases}$$

$$\begin{cases} -392801 - 15432i = (1+2i)^2(1+4i)(4+5i)(2831+924i)(-i) \\ -5479 - 396528i = (1+2i)^2(1+4i)(8+13i)(33+2i)(38+3i) \end{cases}$$

$$\begin{cases} -779326 - 277267i = (1+2i)(1+4i)(2+3i)(10+3i)(29+10i)(41+66i)(-i) \\ 131326 - 1090733i = (1+2i)(1+4i)(8+45i)(1399+2200i) \end{cases}$$

Appendix C

Pairs with all Factors of the Form $4k + 3$

Pairs will all Factors of the form

$4k+3$

$$\begin{cases} 1111259153519361 = 3^4 \cdot 7^2 \cdot 11^2 \cdot 23 \cdot 367 \cdot 467 \cdot 587 \\ 1118172210128127 = 3^4 \cdot 7^2 \cdot 11^2 \cdot 23 \cdot 3023 \cdot 33487 \end{cases}$$

$$\begin{cases} 1334118917120913 = 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 911 \cdot 8447 \\ 1358566300511343 = 3^4 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 1367 \cdot 40127 \end{cases}$$

$$\begin{cases} 3270341090555847 = 3^4 \cdot 7^2 \cdot 11 \cdot 19 \cdot 439 \cdot 587 \cdot 15299 \\ 3281945745924153 = 3^4 \cdot 7^2 \cdot 11 \cdot 19 \cdot 1999 \cdot 1979207 \end{cases}$$

$$\begin{cases} 8062452835794819 = 3^4 \cdot 7^2 \cdot 11^2 \cdot 23 \cdot 71 \cdot 79 \cdot 179 \cdot 727 \\ 8554426893254781 = 3^4 \cdot 7^2 \cdot 11^2 \cdot 103 \cdot 223 \cdot 479 \cdot 1619 \end{cases}$$

$$\begin{cases} 11160803668867083 = 3^4 \cdot 7^2 \cdot 11 \cdot 19 \cdot 419 \cdot 503 \cdot 63839 \\ 11208072889468917 = 3^4 \cdot 7^2 \cdot 11 \cdot 19 \cdot 6299 \cdot 2145023 \end{cases}$$

$$\begin{cases} 11906276468021397 = 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 1087 \cdot 63179 \\ 12117725765249643 = 3^4 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 971 \cdot 503879 \end{cases}$$

$$\begin{cases} 14375494338185673 = 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 743 \cdot 111599 \\ 14642939731916727 = 3^4 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 1619 \cdot 365179 \end{cases}$$

$$\begin{cases} 14435885714987583 = 3^4 \cdot 7^2 \cdot 11 \cdot 19 \cdot 251 \cdot 2243 \cdot 30911 \\ 14499012954908097 = 3^4 \cdot 7^2 \cdot 11 \cdot 19 \cdot 11807 \cdot 1576511 \end{cases}$$

$$\begin{cases} 1767551875449598684552576239 = 3^4 \cdot 7^2 \cdot 11^2 \cdot 31 \cdot 263 \cdot 47 \cdot 424079 \cdot 22648718399 \\ 1805163613534085298922111761 = 3^4 \cdot 7^2 \cdot 11^2 \cdot 31 \cdot 263 \cdot 593711999 \cdot 776527487 \end{cases}$$

$$\begin{cases} 5349260758292158741215099687 = 3^4 \cdot 7^2 \cdot 11^2 \cdot 31 \cdot 79 \cdot 311 \cdot 5879 \cdot 2487566303423 \\ 5367370742012960194122671 3 = 3^4 \cdot 7^2 \cdot 11^2 \cdot 31 \cdot 79 \cdot 82817279 \cdot 55104316847 \end{cases}$$