January 2016

Low-Dimensional Reality-Based Algebras

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LOW DIMENSIONAL REALITY BASED ALGEBRAS

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Submitted to the Faculty of the Graduate School of
Eastern Kentucky University
in partial fulfillment of the requirements
for the degree of
MASTER OF SCIENCE
May 2016

DEDICATION

I dedicate this thesis to all students of the mathematical sciences. May the search for knowledge and understanding never cease.

ACKNOWLEDGEMENTS

I would like to thank my professors from Eastern Kentucky University, Dr. Patrick Costello, Dr. Jeffrey Neugebauer, and Dr. Kirk Jones, who supported and encouraged me during the writing of this thesis. I would also like to thank all the members of the Department of Mathematics and Statistics, especially Dr. Lisa Kay and Dr. Matthew Cropper. I especially want to thank my thesis advisor, Dr. Bangteng Xu, who I worked most closely with during this process. Without his guidance and support, I would not have gotten this far.

I would also like to thank the mathematicians of history who laid the foundations of this work throughout the centuries, as well as all my mathematics teachers and professors who have all made an impact on my love and understanding of the field.

ABSTRACT

In this paper we introduce the definition of a reality-based algebra as well as a subclass of reality-based algebras, table algebras. Using sesquilinear forms, we prove that a reality-based algebra is semisimple. We look at a specific reality-based algebra of dimension 5 of the form $\mathbb{C} \oplus M_2(\mathbb{C})$ and provide formulas for the structure constants of this algebra. We determine by looking at these structure constants and setting conditions on $\delta_1, \delta_2, \delta_3$, and n when this particular reality-based algebra is a table algebra. In fact, this will be a noncommutative table algebra of dimension 5.

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Chapter 1

Definitions and Examples

1.1 Algebras

Definition 1.1. An *algebra* over a field F is a vector space A over F, such that there is a binary operation on A called multiplication (so $x \cdot y \in A$ for each $x, y \in A$), and such that for each $x \in A$, multiplication of the elements of A on the left by x and on the right by x yields linear transformations of A. That is, for all $x, y, z \in A$ and $\lambda \in F$,

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

$$x \cdot (\lambda y) = \lambda(x \cdot y)$$

$$(y + z) \cdot x = y \cdot x + z \cdot x$$

$$(\lambda y) \cdot x = \lambda(y \cdot x).$$

We will sometimes drop the $\cdot$ and write $x \cdot y$ as xy. Note that $x \cdot 0 = 0 = 0 \cdot x$ for all $x \in A$, where 0 is the zero vector.

i. An algebra A is called *associative* if $x(yz) = (xy)z$ for all $x, y, z \in A$.

ii. An algebra A is called *commutative* if $xy = yx$ for all $x, y \in A$.

iii. If there exists an element $1 \in A$ with the property $x \cdot 1 = x = 1 \cdot x$ for all $x \in A$, 1 is called the *multiplicative identity* of A.

Definition 1.2. A *subalgebra* of an algebra A is a vector subspace which is closed under the operation multiplication. So a subalgebra is another algebra, if it has an identity element (either the identity from A or another).

Example 1.3. $\mathbb{C}$ is a two dimensional algebra over its subfield $\mathbb{R}$. More generally, any field is an algebra over itself, as well as an algebra over any subfield.

Example 1.4. Let $F[x]$ be the set of all polynomials over F, where elements are of the form $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ where $a_0, a_1, a_2, ..., a_n \in F$ and $n \in \mathbb{N} \cup \{0\}$. $F[x]$ has two binary operations $+$ and $\cdot$, called addition and multiplication, defined as follows. Let $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ and $g(x) = b_0 + b_1x + b_2x^2 + ... + b_nx^m$, with $m \leq n$. Assume that $a_i = 0$ for any $i > n$ and $b_j = 0$ for any $j > m$. Then define

$$f(x)+g(x) = (a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2+...+(a_m+b_m)x^m+...+(a_n+b_n)x^n,$$

and

$$f(x) \cdot g(x) = c_0 + c_1 + c_2x^2 + ... + c_{m+n}x^{m+n},$$

where

$$c_k = \sum_{i=0}^{k} a_ib_{k-i}, 0 \leq k \leq m + n.$$

and scalar multiplication is defined by

$$\lambda f(x) = \lambda(a_0 + a_1x + a_2x^2 + ... + a_nx^n) = \lambda a_0 + \lambda a_1x + \lambda a_2x^2 + ... + \lambda a_nx^n.$$

Let $\lambda \in F$ and $f(x), g(x), h(x) \in F[x]$, where $f(x)$ is of degree n, $g(x)$ is of degree m, and $h(x)$ is of degree p with $p \leq m \leq n$. $F[x]$ is closed under multiplication since $f(x) \cdot g(x)$ yields a polynomial of degree $n+m$ with coefficients in F. Thus, $f(x) \cdot g(x) \in A$.

$$f(x) \cdot (g(x) + h(x)) = f(x) \cdot g(x) + f(x) \cdot h(x);$$

$$f(x) \cdot (\lambda g(x)) = \lambda(f(x) \cdot g(x));$$

$$(g(x) + h(x)) \cdot f(x) = g(x) \cdot f(x) + h(x) \cdot f(x);$$

$$(\lambda g(x)) \cdot f(x) = \lambda(g(x) \cdot f(x)).$$

Each of these properties follows directly from the commutativity and distributivity of field elements. Thus, each of the conditions specified in the definition is met, and $F[x]$ is an algebra. $F[x]$ is a commutative, associative algebra over F with identity $1 = x^0$, and an infinite basis $\{1 = x^0, x, x^2, ..., x^n, ...\}$ as a vector space over F.

Example 1.5. Let F be a field and $n \in \mathbb{Z}^+$. Let $M_n(F)$ denote the set of all $n \times n$ matrices over a field F, where elements are of the form

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},$$

where each entry $a_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$, is in F. We may simply denote the above matrix as (a_{ij}). The addition of two matrices is defined by

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \dots & a_{nn} + b_{nn} \end{pmatrix},$$

and scalar multiplication is defined by

$$\lambda \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \dots & \lambda a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{n1} & \lambda a_{n2} & \dots & \lambda a_{nn} \end{pmatrix}.$$

If $A = (a_{ij})$ and $B = (b_{ij})$ are each $n \times n$ matrices, the the product AB of A and B is another $n \times n$ matrix whose (i, j)-entry is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Expressing matrix multiplication visually we get

$$AB = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + ... + a_{1n}b_{n1} & a_{11}b_{12} + ... + a_{1n}b_{n2} & \dots & a_{11}b_{1n} + ... + a_{1n}b_{nn} \\ a_{21}b_{11} + ... + a_{2n}b_{n1} & a_{21}b_{12} + ... + a_{2n}b_{n2} & \dots & a_{21}b_{1n} + ... + a_{2n}b_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}b_{11} + ... + a_{nn}b_{n1} & a_{n1}b_{12} + ... + a_{nn}b_{n2} & \dots & a_{n1}b_{1n} + ... + a_{nn}b_{nn} \end{pmatrix}.$$

Let $\lambda \in F$ and $A = (a_{ij}), B = (b_{ij}), C = (c_{ij}) \in M_n(F)$. $M_n(F)$ is closed under multiplication since $A \cdot B$ yields an $n \times n$ matrix with each (a_{ij})-entry in F. Thus, $A \cdot B \in M_n(F)$. With the definitions of matrix addition, scalar multiplication, and matrix multiplication, $M_n(F)$ satisfies all the following properties:

$$A \cdot (B + C) = A \cdot B + A \cdot C$$

$$A \cdot (\lambda B) = \lambda(A \cdot B)$$

$$(B + C) \cdot A = B \cdot A + C \cdot A$$

$$(\lambda B) \cdot A = \lambda(B \cdot A).$$

Thus, the set $M_n(F)$ of all $n \times n$ matrices over F is an algebra under the defined matrix addition and multiplication, and scalar multiplication. $M_n(F)$ is associative, but not commutative (except when n = 1). The identity element is I_n.

Example 1.6. The group algebra $F[G]$, where F is a field and G a group, is the set of all linear combinations of finitely many elements of G with coefficients in F, hence all elements are of the form $a_1g_1 + a_2g_2 + ... + a_ng_n$ where $a_1, ..., a_n \in F$ and $g_1, ..., g_n \in G$. This element can be denoted in general by $\sum\limits_{g\in G} a_g g$. $F[G]$ is an algebra over F with respect to addition defined by

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} (a_g + b_g)g,$$

scalar multiplication defined by

$$\lambda \sum_{g\in G} a_g g = \sum_{g\in G} (\lambda a_g)g,$$

and multiplication defined by

$$\left(\sum_{g\in G} a_g g\right)\left(\sum_{g\in G} b_g g\right) = \sum_{g\in G, h\in G} (a_g b_h)gh.$$

The identity element of G is the unit of $F[G]$ $(1 \cdot e)$. $F[G]$ is associative, and is commutative if and only if G is an Abelian group.

1.2 Association Schemes

Definition 1.7. Let S be a finite set and d a positive integer. For each integer j with $0 \le j \le d$, let R_j be a nonempty relation on S, so that the set of $d + 1$

relations R_j has the following properties:

i. $R_0 = \{(x, x) : x \in S\}$;

ii. $\bigcup_{j=0}^{d} R_j = S \times S$ and $R_i \cap R_j = \emptyset$ if $i \neq j$;

iii. for each j, ${}^tR_j = R_{j^*}$ for some j^* with $0 \leq j^* \leq d$;

iv. for any integers h, i, j from 0 to d, and any pair $(x, y) \in R_j$, the number of elements $z \in S$ such that $(x, z) \in R_h$ and $(z, y) \in R_i$ is independent of the particular choice of x and y. So this nonnegative integer may be denoted by p_{hij}.

Then the configuration $\mathcal{S} := (S, \{R_j\}_{0\leq j\leq d})$ is called an *association scheme* of class d on S. The numbers p_{hij} are called the *intersection numbers* of S. $\mathcal{S}$ is called *commutative* if $p_{hij} = p_{ihj}$ for all h, i, j. Note That $(j^*)^* = j$ for all j. $\mathcal{S}$ is called *symmetric* if $j^* = j$ for all j.

Definition 1.8. Let $\mathcal{S} = (S, \{R_j\}_{0\leq j\leq d})$ be an association scheme. Define the *jth adjacency matrix* A_j of $\mathcal{S}$ to be the matrix of degree $|S|$, whose rows and columns are indexed by the elements of S (in some fixed order), and whose (x, y) entries are

$$(A_j)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_j, \\ 0 & \text{if not.} \end{cases}$$

Note that a given pair (x, y) occurs in just one of the relations R_j, by property *ii.* of Definition 1.7, and hence the (x, y) entry is nonzero (is 1) in just one of the A_j.

Proposition 1.9. *Let* $\mathcal{S} = (S, \{R_j\}_{0\leq j\leq d})$ *be an association scheme, and* $A_0, A_1, ..., A_d$ *the adjacency matrices. Then,*

i. $A_0 = I$;

ii. $\{A_0, A_1, ..., A_d\}$ *is a linearly independent set in* $M_{|S|}(\mathbb{C})$;

iii. $A_0 + A_1 + ... + A_d = J$, *where J is the matrix with all entries equal to 1;*

iv. ${}^tA_j = A_{j^*}$ *(${}^tM$ means the transpose of M, for any matrix M);*

v. $A_hA_i = \sum_{j=0}^{d} p_{hij}A_j$ *for all h, i, j, where the p_{hij} are the intersection numbers for $\mathcal{S}$;*

vi. $p_{hij} = p_{i^*h^*j^*}$*;*

vii. *$\mathcal{S}$ is commutative if and only if $A_hA_i = A_iA_h$ for all h, i;*

viii. *$\mathcal{S}$ is symmetric if and only if ${}^tA_j = A_j$ for all j.*

Proof.

i. A_0 is the adjacency matrix whose (x, y) entries are

$$(A_0)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_0, \\ 0 & \text{if not.} \end{cases}$$

Therefore, the (x, y) entry contains a 1 if and only if $x = y$. Hence, only the diagonal entries contain a 1, and therefore $A_0 = I$.

ii. Suppose that $\lambda_0A_0 + \lambda_1A_1 + ... + \lambda_dA_d = 0$. By property *ii.* of Definition 1.7, we know that $R_i \cap R_j = \emptyset$ if $i \neq j$. Hence, the (x, y) entry is 1 in just one of the adjacency matrices, and all other adjacency matrices have an (x, y) entry of 0. Suppose that the (x, y) entry of A_i is 1 and all other A_j have (x, y) entry 0. Hence, $\lambda_i = 0$ for each i.

iii. The proof for this is straightforward and follows from the fact that $R_i \cap R_j = \emptyset$ if $i \neq j$.

iv. If the (x, y) entry of A_i is 1, then the (y, x) entry of ${}^tA_i$ is 1. ${}^tA_i = A_k$ for some k that depends on i. Hence $k = i^*$, so ${}^tA_i = A_{i^*}$.

v. Suppose the (x, y) entry of A_j is not zero. The (x, y) entry of A_hA_i, $(A_hA_i)_{(x,y)} = \sum_{z \in S}(A_h)_{(x,z)}(A_i)_{(z,y)}$, which represents the number of z such that $(A_h)_{(x,y)} \neq 0$

and $(A_i)_{(x,y)} \neq 0$ with $(x,y) \in A_j$. So, $(A_hA_i)_{(x,y)} = \{z|(x,y) \in R_h; (z,y) \in R_i\} = p_{hij}$. So $A_hA_i = \sum\limits_{j=1}^{d} p_{hij}A_j$.

vi. ${}^t(A_hA_i) = \sum\limits_{j=1}^{d} {}^t(p_{hij}A_j) = A_i^*A_h^* = \sum\limits_{j=1}^{d} p_{hij}A_j^*$. But $A_i^*A_h^* = \sum\limits_{j=1}^{d} p_{h^*i^*j^*}A_j^*$. Hence $p_{hij} = p_{i^*h^*j^*}$.

□

Definition 1.10. Let $\mathcal{S} = (S, \{R_j\}_{0\leq j\leq d})$ be an association scheme, with $|S| = n$. Let $\mathcal{A} = \mathcal{A}(\mathcal{S})$ be the vector subspace of $M_n(\mathbb{C})$ spanned by the adjacency matrices of $\mathcal{S}$. For any two adjacency matrices $A_hA_i = \sum\limits_{j=0}^{d} p_{hij}A_j$ for all h, i, j where the p_{hij} are the intersection numbers for $\mathcal{S}$. From this fact it follows that $\mathcal{A}$ is closed under matrix multiplication, and hence is a subalgebra of $M_n(\mathbb{C})$ which contains $I = A_0$. We call $\mathcal{A}$ the *adjacency algebra*, or the *Bose-Mesner algebra*, of $\mathcal{S}$.

Definition 1.11. Let $\mathcal{S} = (S, R_{j_{0\leq j\leq d}})$ be an association scheme. For each integer i with $0 \leq i \leq d$, define the nonnegative integer k_i as follows: for each $x \in S$,

$$k_i := \text{the number of elements } z \in S \text{ with } (x,z) \in R_i.$$

Then k_i is called the *valency* of R_i.

For any association scheme $\mathcal{S}$, the definitions of k_i and adjacency matrix A_i make it clear that k_i is the sum over any row of A_i.

Proposition 1.12. *Let $\mathcal{S} = (S, \{R_j\}_{0\leq j\leq d}$ be an association scheme with intersection numbers p_{hij}, $0 \leq h, i, j \leq d$, and valencies k_i, $0 \leq i \leq d$. Then the following hold:*

i. $k_i = p_{ii^*0} > 0$, *and k_i is any row sum of A_i;*

ii. $k_0 = 1$;

iii. $k_i = k_{i^*}$;

iv. $p_{hi0} = k_{i^*}\delta_{hi^*}$;

v. $k_h k_i = \sum_{j=0}^{d} p_{hij} k_j$.

Proof.

i. Let $y = x \in S$. Then $(x,x) \in R_0$, and for any $z \in S$, $(x,z) \in R_i$ iff $(z,x) \in R_{i^*}$, by Definition 1.7 (*iii.*). Hence k_i counts the number of $z \in S$ with $(x,z) \in R_i$ and $(z,x) \in R_{i^*}$. However, this means that $k_i = p_{ii^*0}$, which is independent of the choice of $x \in S$. So, $k_i = p_{ii^*0} =$ any row sum of A_i. If $k_i = 0$, then no $x \in S$ is paired with any $z \in S$ in R_i; so R_i would be empty, a contradiction to the definition of an association scheme. Therefore $k_i > 0$ for all i.

ii. Since any $x \in S$ is paired only with itself in R_0, $k_0 = 1$.

iii. If k_i is any row sum of A_i, then $k_i|S| =$ number of entries 1 in $A_i =$ number of entries 1 in ${}^tA_i = k_{i^*}|S|$.

iv. For any $x \in S$, $(x,x) \in R_0$. So, p_{hi0} counts the number of elements $z \in S$ with $(x,z) \in R_h$ and $(z,x) \in R_i$. If $(x,z) \in R_h$, then $(z,x) \in R_{h^*}$, but $(z,x) \in R_i$. So, for $p_{hi0} \neq 0$, $i = h^*$, and $h = i^*$. Thus, if $h \neq i^*$ then $p_{hi0} = 0$. If $h = i^*$, then $p_{hi0} = p_{i^*i0} = k_{i^*}$ by (*i.*).

v. Fix $x \in S$. Then k_h counts the number of elements $z \in S$ with $(x,z) \in R_h$. For each such z, k_i counts the number of elements $u \in S$ with $(z,u) \in R_i$. Hence, $k_h k_i$ is the number of ordered triples (x,z,u) with $(x,z) \in R_h$ and $(z,u) \in R_i$ (for a single fixed x).

Exactly k_j of the elements $u \in S$ are such that $(x,u) \in R_j$. For each such u, there are p_{hij} elements $z \in S$ with $(x,z) \in R_h$ and $(z,u) \in R_i$. So there are exactly $p_{hij}k_j$ of the set of triples above with $(x,u) \in R_j$. Hence there are $\sum_{j=0}^{d} p_{hij}k_j$ of the triples altogether. Since we also calculated the number of these triples as $k_h k_i$, (*v.*) is proved.

□

1.3 Algebra Homomorphisms and Ideals

Definition 1.13. Let A, B be algebras over F. An *algebra homomorphism* $\phi : A \to B$ is a linear transformation such that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$. Then B is called a *homomorphic image* of A if and only if (iff) there is an algebra homomorphism $\phi : A \to B$ which is onto B. A homomorphism is called a *monomorphism* iff it is one-to-one, and is an *isomorphism* from one given algebra to another iff it is both one-to-one and onto. When an isomorphism exists from A onto B, A and B are said to be *isomorphic* ($A \cong B$). An isomorphism of A onto itself is called an *automorphism.*

Since the composition of two homomorphisms is again a homomorphism, it is easily seen that being isomorphic is an equivalence relation among algebras.

Proposition 1.14. *Let A, B be algebras over F, $\mathcal{V} = \{a_i\}_{i=1}^{n}$ a basis for A, and $\phi : A \to B$ a linear transformation.*

i. ϕ is an algebra homomorphism $\Leftrightarrow \phi(a_h a_i) = \phi(a_h)\phi(a_i)$ for all $a_h, a_i \in \mathcal{V}$.

ii. Let $\{b_i\}_{i=1}^{n}$ be a basis for B so that $dimA = dimB$, and let $\{\alpha_{hij}\}, \{\beta_{hij}\}$ be the arrays of n^3 structure constants for $\{a_i\}, \{b_i\}$ respectively. Let $\psi : A \to B$ be the vector space isomorphism such that $\psi(a_i) = b_i$ for $1 \leq i \leq n$. Then

$$\psi \textit{ is an algebra isomorphism} \Leftrightarrow \alpha_{hij} = \beta_{hij} \textit{ for all } h, i, j.$$

Proof.

i. ($\Rightarrow$) Since ϕ is a homomorphism, it follows directly from the definition that $\phi(a_h a_i) = \phi(a_h)\phi(a_i)$ for all $a_h, a_i \in \mathcal{V}$.

($\Leftarrow$) We first need to show that $\phi(x)\phi(y) = \phi(x)\phi(y)$ for all $x, y \in A$. Since

$x, y \in A$, $x = \sum_{i=1}^{n} x_i a_i$ and $y = \sum_{h=1}^{n} y_h a_h$.

$$\begin{aligned}
xy &= \left(\sum_{i=1}^{n} x_i a_i\right)\left(\sum_{h=1}^{n} y_h a_h\right) \\
&= \sum_{i,h} x_i y_h a_i a_h \\
\phi(xy) &= \sum_{i,h} x_i y_h \phi(a_i)\phi(a_h) \\
\phi(x)\phi(y) &= \left(\sum_{i} x_i \phi(a_i)\right)\left(\sum_{h} y_h \phi(a_h)\right) \\
&= \sum_{i,h} x_i y_h \phi(a_i)\phi(a_h) \\
&= \phi(xy).
\end{aligned}$$

Hence, ϕ is an algebra homomorphism.

ii. ($\Rightarrow$) Since $\mathcal{V}$ is a basis for the algebra A, $a_h a_i = \sum_j \alpha_{hij} a_j$ and $b_h b_i = \sum_j \beta_{hij} b_j$. $\psi(a_i) = b_i$ by assumption. So,

$$\begin{aligned}
\sum_j \beta_{hij} b_j \qquad &= b_h b_i &&= \psi(a_h)\psi(a_i) \\
&= \psi(a_h a_i) &&= \psi\left(\sum_j \alpha_{hij} a_j\right) \\
&= \sum_j \alpha_{hij}\psi(a_j) &&= \sum_j \alpha_{hij} b_j
\end{aligned}$$

Therefore we now have $\alpha_{hij} = \beta_{hij}$.

$\square$

Example 1.15. Let $\mathcal{A} = \mathcal{A}(\mathcal{S})$ be the adjacency algebra of an association scheme $\mathcal{S} = (S, \{R_j\}_{0 \le h,i \le d})$, with basis $\{A_i\}_{i=0}^{d}$, where A_i is the adjacency matrix corresponding to the relation R_i. Let $\phi : \mathcal{A} \to \mathbb{C}$ be the linear transformation such that $\phi(A_i) = k_i$ for all i, where $k_i \in \mathbb{Z}^+$ is the valency of the relation R_i. Propositions 1. and 1.12 imply that for all $0 \le h, i \le d$,

$$\phi(A_h A_i) = \phi(\sum_{j=0}^{d} p_{hij} A_j) = \sum_{j=0}^{d} p_{hij}\phi(A_j) = \sum_{j=0}^{d} p_{hij} k_j = k_h k_i = \phi(A_h)\phi(A_i),$$

where the p_{hij} are the intersection numbers for $\mathcal{S}$. Thus, ϕ is an algebra homomorphism by Proposition 1.14(i).

Definition 1.16. Let A be an algebra over F. The *center* of A, denoted $Z(A)$, is the set of those elements of A which commute with all elements of A. That is,

$$Z(A) := \{x \in A | ax = xa, \forall a \in A\}.$$

$Z(A)$ contains 1 and 0, and is a subalgebra of A. Note that $Z(A) = A$ iff A is commutative.

Example 1.17. Let $A = M_n(F)$ then $Z(A)$ is the set of diagonal matrices $Z(A) = \{\alpha I | \alpha \in F\}$. Let (a_{ij}) be an $n \times n$ matrix and I be the $n \times n$ identity matrix. It is known that for a matrix, $(a_{ij})I = I(a_{ij})$.

$$(a_{ij}) \cdot \alpha I = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} \alpha & 0 & \dots & 0 \\ 0 & \alpha & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \dots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \dots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{n1} & \alpha a_{n2} & \dots & \alpha a_{nn} \end{pmatrix}$$

$$= \alpha \cdot \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \dots & 0 \\ 0 & \alpha & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \alpha I(a_{ij}).$$

Definition 1.18. Let A be an algebra over F. A *left ideal* (*right ideal*) C of A is a vector subspace of A such that for all $x \in C$ and $a \in A$, $ax \in C$ ($xa \in C$). An *ideal* of A is a vector subspace which is both a left and a right ideal.

So a right or left ideal C of A is necessarily a subalgebra, but does not contain the multiplicative identity 1_A unless $C = A$. A itself and $\{0\}$ are always ideals of A (called *trivial* ideals). Any other ideals of A are called *proper*.

Proposition 1.19. *Let A be an algebra over a field F and fix any element $y \in A$.*

Define $Ay := \{ay | a \in A\}$ *and* $yA := \{ya | a \in A\}$. *Then* Ay *is a left ideal of* A *and* yA *is a right ideal of* A.

Proof. Let $u, v \in Ay$. Then $u = a_u y$ and $v = a_v y$ for some $a_u, a_v \in A$. So, $u - v = (a_u y) - (a_v y) = (a_u - a_v)y \in Ay$ since $a_u - a_v \in A$. So, Ay is a subspace of A. Let $b \in A$ and $q \in Ay$. Since $q \in Ay$, $q = a_q y$ for some $a_q \in A$. So, $bq = b(a_q y) = (ba_q)y \in Ay$ since algebras are closed under multiplication. So, Ay is a left ideal of A.

The proof for yA is similar. □

Definition 1.20.

i. Let V, W be vector spaces over a field F. Define $Hom_F(V, W)$, the *F-homomorphisms from V into W*, as the set of all linear transformations $\psi : V \to W$.

ii. If C, D are left ideals of an algebra A over F, define $Hom_A(C, D)$, the *A-homomorphisms from C into D* as

$$Hom_A(C, D) = \{\psi | \psi \in Hom_F(C, D) \text{ and } \psi(ac) = a\psi(c) \text{ for all } a \in A, c \in C\}.$$

iii. If C is a left ideal of A, denote $Hom_A(C, C)$ and $End_A(C)$, the *A-endomorphisms of C.*

Definition 1.21. Let C, D be left ideals of an algebra A. Then C and D are called *isomorphic* ($C \cong D$) iff there exists a vector space isomorphism $\psi : C \to D$ with $\psi \in Hom_A(C, D)$. Such ψ is called an *A-isomorphism.*

Example 1.22. Let $A = M_2(F), C = C(1) = \left\{ \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \middle| x, y \in F \right\}, D = C(2) = \left\{ \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \middle| x, y \in F \right\}$. Then $C(1) \cong C(2)$ because the map $\psi : C(1) \to C(2)$

such that $\psi\left(\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}$ is a vector space isomorphism, and for all $\begin{pmatrix} r & s \\ t & u \end{pmatrix} \in A$ and $\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \in C(1)$,

$$\psi\left(\begin{pmatrix} r & s \\ t & u \end{pmatrix}\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}\right) = \psi\left(\begin{pmatrix} rx+sy & 0 \\ tx+uy & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & rs+sy \\ 0 & tx+uy \end{pmatrix}$$
$$= \begin{pmatrix} r & s \\ t & u \end{pmatrix}\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix}\psi\left(\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}\right).$$

Therefore ψ is an A-automorphism.

Definition 1.23. Let C, D be left ideals of an algebra A and $\psi \in Hom_A(C, D)$. Define the *kernel* of ψ as $ker\psi := \{c \in C | \psi(c) = 0\}$; and the *image* of ψ as $\psi(C) := \{\psi(c) | c \in C\}$.

Proposition 1.24. *Let C, D be left ideals of A and $\psi \in Hom_A(C, D)$. Then $ker\psi$ is a left ideal of A contained in C and $\psi(C)$ is a left ideal of A contained in D.*

Proof. By definition, $ker\psi \subseteq C$. $ker\psi \neq \emptyset$ since $0 \in ker\psi$ ($\psi(0) = 0$). Let $x, y \in ker\psi$. $\psi(x - y) = \psi(x) - \psi(y) = 0 - 0 = 0$, so $x - y \in ker\psi$. Let $a \in A$, $k \in ker\psi$. Since $\psi \in Hom_A(C, D), \psi(ak) = a\psi(k) = a \cdot 0 = 0$. So $ak \in ker\psi$. Thus $ker\psi$ is a left ideal of A contained in C.
By definition, $\psi(C) \subseteq D$. $\psi(C) \neq \emptyset$ since $0 \in \psi(C)$ ($\psi(0) = 0$). Let $x, y \in \psi(C)$, so $x = \psi(c_x)$ and $y = \psi(c_y)$ for some $c_x, c_y \in C$. $x - y = \psi(c_x) - \psi(c_y) = \psi(c_x - c_y) \in \psi(C)$. Let $a \in A, d \in \psi(C)$. $d = \psi(c_d)$ for some $c_d \in C$. $ad = a\psi(c_d) = \psi(ac_d)$. Since C is a eft ideal of A, $ac_d \in C$, so $\psi(ac_d) \in \psi(C)$. Thus $\psi(C)$ is a left ideal of A contained in C. $\square$

1.4 Simple Ideals and Semisimple Algebras

Definition 1.25. A nonzero left ideal C of an algebra A is called *simple* if there is no left ideal D with $\{0\} \neq D \subsetneq C$.

Definition 1.26. An algebra A over a field F is called a *direct sum of left ideals* $C_1, C_2, ..., C_n$ if $A = C_1 \oplus C_2 \oplus ... \oplus C_n$ as a vector space that is, each $a \in A$ has a unique representation as

$$a = c_1 + c_2 + ... + c_n, \text{ for some } c_i \in C_i.$$

Definition 1.27. An algebra A over a field F is called *semisimple* if for each left ideal C of A, there exists a left ideal C' such that $A = C \oplus C'$.

Proposition 1.28. *If A is semisimple then each nonzero left ideal of A is a direct sum of simple left ideals. In particular, A is a direct sum of simple left ideals.*

Proof. Let C be a nonzero left ideal of A. If C is simple then the conclusion holds for C.

Assume that there exists a left ideal C_1 of A with $\{0\} \neq C_1 \subsetneq C$. Choosing C_1 to be of minimal dimension, we may assume that C_1 is simple. By hypothesis, there is a left ideal C_1' of A with $A = C_1 \oplus C_1'$. Then $C_1' \cap C$ is a left ideal of A. For any $x \in C, x = c_1 + c_1'$ for some $c_1 \in C_1$ and $c_1' \in C_1'$. Since $C_1 \subseteq C, c_1' = x - c_1 \in C_1' \cap C$. It follows that $C = C_1 + (C_1' \cap C)$. Since $C_1 \cap (C_1' \cap C) \subseteq C_1 \cap C_1' = \{0\}$, we have $C = C_1 \oplus (C_1' \cap C)$. By induction on dimension, we may assume that $C_1' \cap C = C_2 \oplus ... \oplus C_m$, a direct sum of simple left ideals of A. Therefore, $C = C_1 \oplus C_2 \oplus ... \oplus C_m$. □

Definition 1.29. An *anti-automorphism* of an algebra A over a field is a vector space isomorphism $\sigma : A \to A$ such that $\sigma(xy) = \sigma(y)\sigma(x)$ for all $a, y \in A$. In other words, σ is an algebra isomorphism between A and A^{op}, where A^{op} is the *opposite algebra* of A defined as the same vector space as A, but with multiplication $*$ defined by $a * b = ba, \forall a, b \in A$.

If A is commutative, any automorphism of A is an anti-automorphism. If $A = M_n(F)$, then the matrix transpose map is an anti-automorphism. If $A \ncong A^{op}$ then of course A has no anti-automorphism.

Theorem 1.30. ***(Wedderburn-Artin Theorem)*** *Let R be a ring satisfying the descending chain condition on left ideals and with no two-sided ideals except* (0) *and R. Then there exists a positive integer n such that for any minimal non-trivial left ideal L of R the following hold:*

i. *R is isomorphic to a direct sum of n copies of L.*

ii. *If $D = End_R(L)$ then D is a skew field and L is an n-dimensional vector space over D.*

iii. $R \approx End_D(L)$.

Chapter 2

Reality-Based Algebras

Definition 2.1. A *reality-based algebra* (RBA) (A, B) is an algebra A over $\mathbb{C}$ with a distinguished basis $B = \{b_i | 0 \leq i \leq d\}$, where $d < \infty$, $b_0 = 1_A$, and the following three conditions hold:

i. For all $0 \leq i, j \leq d$,

$$b_i b_j = \sum_{l=0}^{d} \beta_{ijl} b_l,$$

where each coefficient (structure constant) β_{ijl} is in $\mathbb{R}$.

ii. There is an algebra anti-automorphism $*$ of A, such that $(*)^2 = id_A$ and $B^* = B$. (So $*$ has order at most two, and permutes the elements of B. Set $b_{i*} := b_i^*$.)

iii. For all $0 \leq i, j \leq d$,

$$\beta_{ij0} = 0 \text{ if } j \neq i^*, \text{ and } \beta_{ii*0} = \beta_{i*i0} > 0.$$

Definition 2.2. A *degree map* δ is an algebra homomorphism from A to $\mathbb{C}$ such that $\delta(b) \in \mathbb{R} \setminus \{0\}$ for all $b \in B$. If $\delta(b) > 0$ for all $b \in B$, then δ is called a *positive degree map*.

Definition 2.3.

i. A *C-algebra* is a commutative reality-based algebra with a degree map.

ii. A *table algebra* is a reality-based algebra with all nonnegative structure constants.

Proposition 2.4. *A table algebra has a unique positive degree map.*

The proof of this property requires the orthogonality relations of irreducible characters of table algebras. For a proof see [4].

Example 2.5. Given the basis $\{b_0, b_1, b_2, b_3, b_4\}$ where

$$b_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b_1 = \begin{pmatrix} 6 & 0 & 0 \\ 0 & \frac{-1+5\sqrt{3}}{4} & 0 \\ 0 & 0 & \frac{-1-5\sqrt{3}}{4} \end{pmatrix}, b_2 = \begin{pmatrix} 6 & 0 & 0 \\ 0 & \frac{-3-5\sqrt{3}}{12} & \frac{5}{\sqrt{6}} \\ 0 & \frac{5}{\sqrt{6}} & \frac{-3+5\sqrt{3}}{12} \end{pmatrix},$$

$$b_3 = \begin{pmatrix} 6 & 0 & 0 \\ 0 & \frac{-3-5\sqrt{3}}{12} & \frac{-5\sqrt{6}+3\sqrt{2}}{12} \\ 0 & \frac{-5\sqrt{6}-3\sqrt{2}}{12} & \frac{-3+5\sqrt{3}}{12} \end{pmatrix}, b_4 = \begin{pmatrix} 6 & 0 & 0 \\ 0 & \frac{-3-5\sqrt{3}}{12} & \frac{-5\sqrt{6}-3\sqrt{2}}{12} \\ 0 & \frac{-5\sqrt{6}+3\sqrt{2}}{12} & \frac{-3+5\sqrt{3}}{12} \end{pmatrix}.$$

i. The structure constants for the algebra elements are as follows:

$$b_0b_j \rightarrow \begin{cases} \beta_{0jl} = 1 & \text{if } l = j \\ \beta_{0jl} = 0 & \text{if } l \neq j, \end{cases} \quad b_ib_0 \rightarrow \begin{cases} \beta_{i0l} = 1 & \text{if } l = i \\ \beta_{i0l} = 0 & \text{if } l \neq i, \end{cases}$$

$b_1b_1 \rightarrow \{\beta_{110} = 6, \beta_{111} = \frac{7}{8}, \beta_{112} = \frac{11}{8}, \beta_{113} = \frac{11}{8}, \beta_{114} = \frac{11}{8}\}.$

$b_1b_2 \rightarrow \{\beta_{120} = 0, \beta_{121} = \frac{11}{8}, \beta_{122} = \frac{11}{8}, \beta_{123} = \frac{23}{8}, \beta_{124} = \frac{3}{8}\}.$

$b_1b_3 \rightarrow \{\beta_{130} = 0, \beta_{131} = \frac{11}{8}, \beta_{132} = \frac{23}{8}, \beta_{133} = \frac{1}{8}, \beta_{134} = \frac{13}{8}\}.$

$b_1b_4 \rightarrow \{\beta_{140} = 0, \beta_{141} = \frac{11}{8}, \beta_{142} = \frac{3}{8}, \beta_{143} = \frac{13}{8}, \beta_{144} = \frac{21}{8}\}.$

$b_2b_1 \rightarrow \{\beta_{210} = 0, \beta_{211} = \frac{11}{8}, \beta_{212} = \frac{11}{8}, \beta_{213} = \frac{3}{8}, \beta_{214} = \frac{23}{8}\}.$

$b_2b_2 \rightarrow \{\beta_{220} = 6, \beta_{221} = \frac{11}{8}, \beta_{222} = \frac{7}{8}, \beta_{223} = \frac{11}{8}, \beta_{224} = \frac{11}{8}\}.$

$b_2b_3 \rightarrow \{\beta_{230} = 0, \beta_{231} = \frac{3}{8}, \beta_{232} = \frac{11}{8}, \beta_{233} = \frac{21}{8}, \beta_{234} = \frac{13}{8}\}.$

$b_2b_4 \rightarrow \{\beta_{240} = 0, \beta_{241} = \frac{23}{8}, \beta_{242} = \frac{11}{8}, \beta_{243} = \frac{13}{8}, \beta_{244} = \frac{1}{8}\}.$

$b_3b_1 \rightarrow \{\beta_{310} = 0, \beta_{311} = \frac{11}{8}, \beta_{312} = \frac{3}{8}, \beta_{313} = \frac{21}{8}, \beta_{314} = \frac{13}{8}\}.$

$b_3b_2 \rightarrow \{\beta_{320} = 0, \beta_{321} = \frac{23}{8}, \beta_{322} = \frac{11}{8}, \beta_{323} = \frac{1}{8}, \beta_{324} = \frac{13}{8}\}.$

$b_3b_3 \rightarrow \{\beta_{330} = 0, \beta_{331} = \frac{13}{8}, \beta_{332} = \frac{13}{8}, \beta_{333} = \frac{9}{8}, \beta_{334} = \frac{13}{8}\}.$

$b_3b_4 \to \{\beta_{340} = 6, \beta_{341} = \frac{1}{8}, \beta_{342} = \frac{21}{8}, \beta_{343} = \frac{9}{8}, \beta_{344} = \frac{9}{8}\}$.

$b_4b_1 \to \{\beta_{410} = 0, \beta_{411} = \frac{11}{8}, \beta_{412} = \frac{23}{8}, \beta_{413} = \frac{13}{8}, \beta_{414} = \frac{1}{8}\}$.

$b_4b_2 \to \{\beta_{420} = 0, \beta_{421} = \frac{3}{8}, \beta_{422} = \frac{11}{8}, \beta_{423} = \frac{13}{8}, \beta_{424} = \frac{21}{8}\}$.

$b_4b_3 \to \{\beta_{430} = 6, \beta_{431} = \frac{21}{8}, \beta_{432} = \frac{1}{8}, \beta_{433} = \frac{9}{8}, \beta_{434} = \frac{9}{8}\}$.

$b_4b_4 \to \{\beta_{440} = 0, \beta_{441} = \frac{13}{8}, \beta_{442} = \frac{13}{8}, \beta_{443} = \frac{13}{8}, \beta_{444} = \frac{9}{8}\}$.

ii. Define $* : A \to A, b_i \mapsto b_i^T = b_i^*$ (the algebra anti-automorphism is defined as the transpose of the matrix). $b_0^* = b_0, b_1^* = b_1, b_2^* = b_2, b_3^* = b_4, b_4^* = b_3$. Notice that $*$ has order two, and it permutes the basis elements.

iii. Looking at the structure constants, $\beta_{ij0} = 0$ when $j \neq i^*$. $\beta_{00^*0} = \beta_{000} = \beta_{0^*00} = 6 > 0$; $\beta_{11^*0} = \beta_{110} = \beta_{1^*10} = 6 > 0$; $\beta_{22^*0} = \beta_{220} = \beta_{2^*20} = 6 > 0$; $\beta_{33^*0} = \beta_{340} = \beta_{4^*40} = 6 = \beta_{3^*30} = \beta_{430} = \beta_{44^*0} > 0$.

So, this is a reality-based algebra of dimension 5. In fact, it is a noncommutative table algebra of dimension 5 with degree map $\delta(b_0) = 1$ and $\delta(b_1) = \delta(b_2) = \delta(b_3) = \delta(b_4) = 6$.

Example 2.6. The adjacency algebra is an example of an RBA. The set of *adjacency matrices* forms a basis for the algebra that it generates, $B = \{A_0, A_1, ..., A_d\}$. By Proposition 1.9, $A_0 = I$; the matrix transpose map permutes the elements of B and thus is an anti-automorphism of $\mathcal{A}$ whose square is $id_{\mathcal{A}}$; and the structure constants p_{hij} for B are nonnegative integers. This satisfies Definition 2.1(*i.*) and 2.1(*ii.*).

2.1(*iii.*) is satified by the fact that $p_{ij0} = 0$ if $j \neq i^*$, and $p_{ii^*0} = k_i > 0$, where k_i is the sum over each row of the $n \times n$ matrix A_i, and k_{i^*} is the sum over each column.

Proof. If $x \in S, (x, x) \in R_0$. Hence, p_{hi0} counts the number of elements $z \in S$ with $(x, z) \in R_h$ and $(z, x) \in R_i$. Since $(x, z) \in R_h, (z, x) \in R_{h^*}$. Since $R_i \cap R_j = \emptyset$ if $i \neq j$, $h^* = i$ and $i^* = h$. Hence, $p_{hi0} = p_{i^*i0} = k_{i^*}$. From Proposition 1.9 we know that $p_{hij} = p_{i^*h^*j^*}$, and so $p_{hi0} = p_{i^*i0} = p_{ii^*0}$. It follows also that if $h \neq i^*$, then $p_{hi0} = 0$. $\square$

And so Definition 2.1(*iii.*) is satisfied and an adjacency algebra is an RBA.

2.1 Sesquilinear Forms

Definition 2.7. Let V be a vector space over $\mathbb{C}$. A *sesquilinear form on V* is a function $S : V \times V \to \mathbb{C}$ (where $S(u, v)$ will denote the complex number assigned to the pair of vectors (u, v)), such that the following properties hold for all $u, v, w \in V$ and $\alpha \in \mathbb{C}$:

$$S(u, v + w) = S(u, v) + S(u, w)$$

$$S(\alpha u, v) = \alpha \cdot S(u, v)$$

$$S(v, u) = \overline{(S(u, v))}.$$

Remark: $\overline{\alpha}$ denotes the complex conjugate of α.

Proposition 2.8. *Let S be a sesquilinear form on a vector space V over $\mathbb{C}$. Then, for all $u, v, w \in V$ and $\alpha \in \mathbb{C}$,*

i. $S(v + w, u) = S(v, u) + S(w, u)$;

ii. $S(u, \alpha v) = \overline{\alpha} \cdot S(u, v)$;

iii. $S(v, 0) = 0 = S(0, v)$;

iv. $S(v, v) \in \mathbb{R}$.

Proof.

i. $S(v + w, u) = \overline{(S(u, v + w))} = \overline{(S(u, v) + S(u, w))} = \overline{(S(u, v))} + \overline{(S(u, w))} = S(v, u) + S(w, u)$.

ii. $S(u, \alpha v) = \overline{(S(\alpha v, u))} = \overline{(\alpha S(v, u))} = (\overline{\alpha})\overline{(S(v, u))} = \overline{\alpha} S(u, v)$.

iii. For a vector $u \in V$ $S(v,0) = S(v,0\cdot u) = \overline{0}S(v,u) = 0\cdot S(v,u) = 0 = 0\cdot S(u,v) = S(0\cdot u, v) = S(0,v)$.

iv. $S(v,v) = \overline{(S(v,v))}$, which is only true when $S(v,v) \in \mathbb{R}$.

□

Definition 2.9. A sesquilinear form S on V is called *positive definite* (*nonnegative definite*) if $S(v,v) > 0$ ($S(v,v) \geq 0$) for all $v \neq 0 \in V$.

Definition 2.10. Let S be a sesquilinear form on V, and let $B = \{v_i\}$ be a basis for V over $\mathbb{C}$. Then B is called *orthogonal (with respect to S)* if $S(v_i, v_j) = 0$ for all $i \neq j$. B is called *orthonormal* if B is orthogonal and $S(v_i, v_i) = 1$ for all $v_i \in B$.

Example 2.11. Let vector space V over $\mathbb{C}$ be finite dimensional, with basis $B = \{v_1, ..., v_n\}$. Fix real numbers $\beta_1, \beta_2, ..., \beta_n$. Each $v \in V$ is written uniquely as

$$v = \alpha_1(v)v_1 + \alpha_2(v)v_2 + ... + \alpha_n(v)v_n,$$

where $\alpha_1(v), \alpha_2(v), ..., \alpha_n(v)$ are complex numbers. Define $S : V \times V \to \mathbb{C}$ by

$$S(u,v) = \sum_{i=1}^{n} \beta_i \alpha_i(u)\overline{\alpha_i(v)}.$$

Then S is a sesquilinear form on V, and B is orthogonal with respect to S. S is positive definite iff all $\beta_i > 0$, and S is nonnegative definite iff all $\beta_i \geq 0$. If S is positive definite, the basis $\left\{\frac{1}{\sqrt{\beta_i}}v_i : v_i \in B\right\}$ is orthonormal with respect to S.

Proposition 2.12. *(Gram-Schmidt Orthonormalization) Let vector space V over $\mathbb{C}$ have basis $\{v_i\}_{i=1}^n$. Let S be a positive definite sesquilinear form on V. Then one can obtain $B = \{u_i\}_{i=1}^n$, and orthonormal basis with respect to S, such that for all $1 \leq j \leq n$, $\langle u_1, u_2, ..., u_j\rangle = \langle v_1, v_2, ..., v_j\rangle$.*

Proof. Let $\beta_1 = S(v_1, v_1) > 0$, and let $u_1 = \frac{1}{\sqrt{\beta_1}}v_1$. Then $S(u_1, u_1) = 1$ and $\langle u_1\rangle = \langle v_1\rangle$.

Suppose inductively that $u_1, ..., u_t$ are vectors in V such that $\langle u_1, ..., u_t \rangle = \langle v_1, ..., v_t \rangle$, and $S(u_i, u_j) = \delta_{ij}$ for all $1 \leq i, j \leq t$. Define

$$w := v_{t+1} - \sum_{i=1}^{t} S(v_{t+1}, u_i) u_i.$$

So, $w \neq 0$, $S(w, u_j) = 0$ for all $1 \leq j \leq t$, and $\langle u_1, ..., u_t, w \rangle = \langle v_1, ..., v_t, v_{t+1} \rangle$. Let $\beta = S(w, w) > 0$, and define $u_{t+1} = \frac{1}{\sqrt{\beta}} w$. It follows that $S(u_{t+1}, u_j) = 0 = S(u_j, u_{t+1})$ for all $1 \leq j \leq t, S(u_{t+1}, u_{t+1}) = 1$, and $\langle u_1, ..., u_t, u_{t+1} \rangle = \langle v_1, ..., v_t, v_{t+1} \rangle$. So by induction on t, $B = \{u_i\}_{i=1}^n$ can be obtained as desired. $\square$

Proposition 2.13. *Let S be a positive definite sesqilinear form on a finite dimensional vector space V over $\mathbb{C}$ Let U be any subspace of V. Then $U \cap U^\perp = \{0\}$ and $U + U^\perp = V$. In other words, $V = U \oplus U^\perp$, so that the union of any bases of U and $U^\perp$ is a basis of V.*

Proof. If $u \neq 0$ is in U, then $S(u, u) > 0$ implies that $u \notin U^\perp$. Hence, $U \cap U^\perp = \{0\}$. Let $\{w_1, ..., w_m, v_{m+1}, ..., v_n\}$ be a basis for V such that $\{w_1, ..., w_m\}$ is a basis for U. Application of the Gram-Schmidt process to this basis yields a basis $\{u_1, ...u_m, u_{m+1}, ..., u_n\}$ for V wich is an orhtonormal basis for V with espect to S, and so that $\langle u_1, ..., u_m \rangle = \langle w_1, ..., w_m \rangle = U$. Now for each $j > m$ and $i \leq m$, $S(u_j, u_i) = 0$ implies that $u_j \in U^\perp$. Hence $\langle u_{m+1}, ..., u_n \rangle \subseteq U^\perp$ and we have $U + Y^\perp = V$. (In fact $V = U \oplus U^\perp$ yields dim$U$+dim$U^\perp$ =dim$V = n$, hence $U^\perp = \langle u_{m+1}, ..., u_n \rangle$.) $\square$

2.2 Reality-Based Algebras are Semisimple.

Theorem 2.14. *Suppose that A is an algebra over $\mathbb{C}$ and that S is a positive definite sesquilinear form on A with the property that for all $x, y, z \in A$ there exists some $\hat{x} \in A$ such that $S(xy, z) = S(y, \hat{x}z)$. Then A is semisimple.*

Proof. Let C be any left ideal of A. By Proposition 2.13, $A = C \oplus C^\perp$ as a direct sum of vector subspaces, where $C^\perp$ is the subspace of all vectors $v \in A$ such that

$S(v,c)=0$ for all $c \in C$. By definition of semisimple, it suffices to prove that $C^{\perp}$ is a left ideal. For all $v \in C^{\perp}, a \in A$ and $c \in C, S(av,c) = S(v,\hat{a}c)$ for some $\hat{a} \in A$, by hypothesis. Since C is a left ideal, $\hat{a}c \in C$. Hence, $S(v,\hat{a}c) = 0$. Then $S(av,c) = 0$ for all $c \in C$, whence $av \in C^{\perp}$. The result follows. $\square$

Lemma 2.15. *Let (A,B) be reality-based.*

i. When writing elements of A as linear combinations of B, the coefficient of b_0 in xy equals the coefficient of b_0 in yx, for all $x, y \in A$.

ii. For all indices i, j, t, $\beta_{tt^*0}\beta_{ijt} = \beta_{jj^*0}\beta_{i^*tj}$.

Proof.

i. Let $x = \sum\limits_{i=0}^{d} \alpha_i b_i, y = \sum\limits_{i=0} \gamma_i b_i$, for some $\alpha_i, \gamma_i \in \mathbb{C}$. Then $xy = \sum\limits_{i,j} \alpha_i \gamma_i b_i b_j$ implies that the coefficient of b_0 in xy equals

$$\sum_{i,j} \alpha_i \gamma_j \beta_{ij0} = \sum_{i^*} \gamma_{i^*} \alpha_i \beta_{i^*i0},$$

by Definition 2.1. Similarly, the coefficient of b_0 in yx equals

$$\sum_{i,j} \gamma_i \alpha_j \beta_{ij0} = \sum_{i} \gamma_i \alpha_{i^*} \beta_{ii^*0} = \sum_{i^*} \gamma_{i^*} \alpha_i \beta_{i^i0}.$$

Since $\beta_{ii^*0} = \beta_{i^*i0}$ by Definition 2.1, the result follows.

ii. Since $(b_i b_j) b_{t^*} = \sum\limits_{m=0}^{d} \beta_{ijm} b_m b_{t^*}$, the coefficient of b_0 in $(b_i b_j) b_{t^*}$ is $\beta_{ijt}\beta_{tt^*0}$ by Definition 2.1. Since $(b_i b_j) b_{t^*} = b_i(b_j b_{t^*})$, *i.* implies that b_0 has the same coefficient in $(b_j b_{t^*}) b_i$. Since $*$ permutes the elements of the basis, B, and $b_0^* = b_0, b_0$ has the same coefficient in $(b_j b_{t^*} b_i)^* = b_{i^*} b_t b_{j^*}$. Since $b_{i^*} b_t b_{j^*} = \sum\limits_{m=0}^{d} \beta_{i^*tm} b_m b_{j^*}$, this coefficient is $\beta_{i^*tj}\beta_{jj^*0}$. Thus, $\beta_{tt^*0}\beta_{ijt} = \beta_{jj^*0}\beta_{i^*tj}$.

$\square$

Definition 2.16. Let $x \in A$, where (A,B) is reality-based, and set $x = \sum\limits_{i=0}^{d} \alpha_i b_i$, for $\alpha_i \in \mathbb{C}$. Define $\hat{x} = \sum\limits_{i=0}^{d} \overline{\alpha_i} b_i^*$.

Lemma 2.17. *Let (A, B) be reality-based. There exists a positive definite sesquilinear form S on A such that B is orthogonal with respect to S and $S(b_i, b_i) = \beta_{ii^*0}$ for $0 \leq i \leq d$. Furthermore, for all $x, y, z \in A$,*

$$S(xy, z) = S(y, \hat{x}z).$$

Proof. For any $u, v \in A$, let $u = \sum_{i=0}^{d} \alpha_i b_i, v = \sum_{i=0}^{d} \gamma_i b_i$, with $\alpha_i, \gamma_i \in \mathbb{C}$. Define

$$S(u, v) = \sum_{i=0}^{d} \beta_{ii^*0} \alpha_i \overline{\gamma_i}.$$

Then, as in Example 2.11, S is a positive definite sesquilinear form on A for which B is orthogonal. The definition implies that $S(b_i, b_i) = \beta_{ii^*0}$ for all i. For all $0 \leq i, j, t \leq d$,

$$\begin{aligned}
S(b_i b_j, b_t) &= S\left(\sum_m \beta_{ijm} b_m, b_t\right) = \sum_m \beta_{ijm} s(b_m, b_t) \\
&= \beta_{ijt} S(b_t, b_t) \text{ (B is orthogonal)} \\
&= \beta_{ijt} \beta_{tt^*0} = \beta_{i^*tj} \beta_{jj^*0} \text{ (Lemma 2.15 (ii))} \\
&= S(b_j, \beta_{i^*tjb_j}) \text{ (since $\beta_{i^*tj} = \overline{\beta_{i^*tj}}$ (Definition 2.1))} \\
&= \sum_m S(b_j, \beta_{i^*tm} b_m) \text{ (B is orthogonal)} \\
&= S(b_j, \sum_m \beta_{i^*tm} b_m) = S(b_j, b_{i^*} b_t)
\end{aligned}$$

So $S(b_i b_j, b_t) = S(b_j, b_{i^*} b_t$ for all i, j, t.

For any $x, y, z \in A$, set $x = \sum_i \alpha_i b_i, y = \sum_i \beta_i b_i, z = \sum_i \gamma_i z_i$ with $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$. Then

$$\begin{aligned}
S(xy, z) &= S\left(\sum_{i,j} \alpha_i \beta_j b_i b_j, \sum_t \gamma_t b_t\right) \\
&= \sum_{i,j,t} \alpha_i \beta_j \overline{\gamma_t} S(b_i b_j, b_t) = \sum_{i,j,t} \alpha_i \beta_j \overline{\gamma_t} S(b_j, b_{i^*} b_t) \\
&= \sum_{i,j,t} S(\beta_j b_j, \overline{\alpha_i} \gamma_t b_{i^*} b_t) \\
&= S\left(\sum_j \beta_j b_j, \left(\sum_i \overline{\alpha_i} b_{i^*}\right)\left(\sum_t \gamma_t b_t\right)\right) = S(y, \hat{x}z).
\end{aligned}$$

□

Theorem 2.18. *If an algebra A is reality-based, then A is semisimple.*

Proof. Since A is reality-based, Lemma 2.17 tells us that there is a positive definite sesqulinear form S on A such that the basis is orthogonal and $S(b_i, b_i) = \beta_{ii^*0}$ for $0 \leq i \leq d$. Also, for all $x, y, z \in A$, $S(xy, z) = S(y, \hat{x}z)$. It follows directly from Theorem 2.14 that A is semisimple. $\square$

2.3 Noncommutative Reality-Based Algebras of Dimension 5

Herman, Muzychuk, and Xu show in [1] that a finite-dimensional noncommutative semisimple algebra with involution always has an RBA-basis. They focus on algebras of the form $C \oplus M_n(\mathbb{C}), n \geq 2$ and ask questions involving whether or not the RBA admits a positive degree map. For RBAs that have a positive degree map, they try to find an RBA-basis with nonnegative structure constants to determine if there is a generalized table algebra structure. Below we list the key theorems proven in [1].

Theorem 2.19. *For all $n > 2$, $M_n(\mathbb{C})$ with the conjugate-transpose involution has a rational RBA-basis.*

Theorem 2.20. *Let $\mathbb{C}C_2$ be the complex group algebra of the group $C_2 = \{e, x\}$, and let δ be the trivial character of the group C_2. Let $\mathbf{B}$ be a rational RBA-basis of $M_n(\mathbb{C})$.*

Then $C_2 \bigcirc_\delta \mathbf{B}$ is a rational RBA-basis of $\mathbb{C} \oplus M_n(\mathbb{C})$.

Theorem 2.21. *Let $(A, \mathbf{B})$ be an integral RBA with a positive degree map.*

If $|Irr(A)| = 2$, then $|\mathbf{B}| = 2$.

The following theorem was also proven in [1], and is the key theorem we focus on for the work presented in the rest of this paper.

Theorem 2.22. *Suppose*

$$\{(1, I), (\delta_1, \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}), (\delta_2, \begin{bmatrix} v & w \\ w & x \end{bmatrix}), (\delta_3, \begin{bmatrix} r & s \\ t & u \end{bmatrix}, (\delta_3, \begin{bmatrix} r & t \\ s & u \end{bmatrix})\}$$

is a standardized RBA$^\delta$-basis of $\mathbb{C} \oplus M_2(\mathbb{C})$ with respect to the conjugate-transpose involution. Assume all matrix entries of these basis elements are real numbers. Let $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1$ be three sign choices. Then the matrix entries satisfy the identities

$$\begin{aligned}
a &= -\tfrac{\delta_1}{n-1} + \varepsilon_1 \tfrac{\sqrt{n\delta_1(n-1-\delta_1)}}{n-1}, \\
d &= -\tfrac{\delta_1}{n-1} + \varepsilon_1 \tfrac{\sqrt{n\delta_1(n-1-\delta_1)}}{n-1}, \\
v &= -\tfrac{\delta_2}{n-1} - \varepsilon_1 \tfrac{n\delta_1\delta_2}{(n-1)\sqrt{n\delta_1(n-1-\delta_1)}}, \\
x &= -\tfrac{\delta_2}{n-1} + \varepsilon_1 \tfrac{n\delta_1\delta_2}{(n-1)\sqrt{n\delta_1(n-1-\delta_1)}}, \\
w &= \varepsilon_2 \sqrt{\tfrac{2\delta_2\delta_3}{(n-1)(n-1-\delta_1)}}, \\
r &= -\tfrac{\delta_3}{n-1} - \varepsilon_1 \tfrac{n\delta_1\delta_3}{(n-1)\sqrt{n\delta_1(n-1-\delta_1)}}, \\
u &= -\tfrac{\delta_3}{n-1} + \varepsilon_1 \tfrac{n\delta_1\delta_3}{(n-1)\sqrt{n\delta_1(n-1-\delta_1)}}, \\
s &= -\tfrac{w}{2} + \varepsilon_3 \sqrt{\tfrac{\delta_3 n}{2(n-1)}}, \quad \textit{and} \\
t &= -\tfrac{w}{2} - \varepsilon_3 \sqrt{\tfrac{\delta_3 n}{2(n-1)}}.
\end{aligned}$$

Conversely, given positive real numbers n, δ_1, δ_2, and δ_3 satisfying $n = 1 + \delta_1 + \delta_2 + 2\delta_3$ and three choices of sign for ε_1, ε_2, and ε_3, the above identities produce an RBA$^\delta$-basis of $\mathbb{C} \oplus M_2(\mathbb{C})$ having real matrix entries.

Also, formulas for the structure constants for the basis were calculated. The formulas for these structure constants are (with $\varepsilon := \varepsilon_1\varepsilon_2\varepsilon_3$, and not including those involving b_0):

$$
\begin{aligned}
\lambda_{111} &= \tfrac{(n+1)\delta_1^2-3(n-1)\delta_1}{(n-1)^2},\\
\lambda_{112}=\lambda_{113}=\lambda_{114} &= \tfrac{(n+1)\delta_1^2-(n-1)\delta_1}{(n-1)^2},\\
\lambda_{121}=\lambda_{211} &= \tfrac{(n+1)\delta_1\delta_2-(n-1)\delta_2}{(n-1)^2},\\
\lambda_{122}=\lambda_{212} &= \tfrac{(n+1)\delta_1\delta_2-(n-1)\delta_1}{(n-1)^2},\\
\lambda_{123}=\lambda_{214} &= \tfrac{(n+1)\delta_1\delta_2+\varepsilon(n-1)\sqrt{n\delta_1\delta_2}}{(n-1)^2},\\
\lambda_{124}=\lambda_{213} &= \tfrac{(n+1)\delta_1\delta_2-\varepsilon(n-1)\sqrt{n\delta_1\delta_2}}{(n-1)^2},\\
\lambda_{131}=\lambda_{141}=\lambda_{311}=\lambda_{411} &= \tfrac{(n+1)\delta_1\delta_3-(n-1)\delta_3}{(n-1)^2},\\
\lambda_{132}=\lambda_{412} &= \tfrac{(n+1)\delta_1\delta_2\delta_3+\varepsilon(n-1)\delta_3\sqrt{n\delta_1\delta_2}}{\delta_2(n-1)^2},\\
\lambda_{133}=\lambda_{414} &= \tfrac{(n+1)\delta_1\delta_3-(n-1)\delta_1-\varepsilon(n-1)\sqrt{n\delta_1\delta_2}}{(n-1)^2},\\
\lambda_{134}=\lambda_{143}=\lambda_{314}=\lambda_{413} &= \tfrac{(n+1)\delta_1\delta_3}{(n-1)^2}\\
\lambda_{142}=\lambda_{312} &= \tfrac{(n+1)\delta_1\delta_2\delta_3-\varepsilon(n-1)\delta_3\sqrt{n\delta_1\delta_2}}{\delta_2(n-1)^2},\\
\lambda_{144}=\lambda_{313} &= \tfrac{(n+1)\delta_1\delta_3-(n-1)\delta_1+\varepsilon(n-1)\sqrt{n\delta_1\delta_2}}{(n-1)^2},\\
\lambda_{221}=\lambda_{223}=\lambda_{224} &= \tfrac{(n+1)\delta_2^2-(n-1)\delta_2}{(n-1)^2},\\
\lambda_{222} &= \tfrac{(n+1)\delta_2^2-3(n-1)\delta_2}{(n-1)^2},\\
\lambda_{231}=\lambda_{421} &= \tfrac{(n+1)\delta_1\delta_2\delta_3-\varepsilon(n-1)\delta_3\sqrt{n\delta_1\delta_2}}{\delta_1(n-1)^2},\\
\lambda_{232}=\lambda_{422}=\lambda_{242}=\lambda_{322} &= \tfrac{(n+1)\delta_2\delta_3-(n-1)\delta_3}{(n-1)^2},\\
\lambda_{233}=\lambda_{424} &= \tfrac{(n+1)\delta_2\delta_3-(n-1)\delta_2+\varepsilon(n-1)\sqrt{n\delta_1\delta_2}}{(n-1)^2},\\
\lambda_{234}=\lambda_{423}=\lambda_{243}=\lambda_{324} &= \tfrac{(n+1)\delta_2\delta_3}{(n-1)^2},\\
\lambda_{241}=\lambda_{321} &= \tfrac{(n+1)\delta_1\delta_2\delta_3+\varepsilon(n-1)\delta_3\sqrt{n\delta_1\delta_2}}{\delta_1(n-1)^2},\\
\lambda_{244}=\lambda_{323} &= \tfrac{(n+1)\delta_2\delta_3-(n-1)\delta_2-\varepsilon(n-1)\sqrt{n\delta_1\delta_2}}{(n-1)^2},\\
\lambda_{331}=\lambda_{332}=\lambda_{334} &= \tfrac{(n+1)\delta_3^2}{(n-1)^2},\\
\lambda_{441}=\lambda_{442}=\lambda_{443} &= \tfrac{(n+1)\delta_3^2}{(n-1)^2},\\
\lambda_{343}=\lambda_{344}=\lambda_{433}=\lambda_{434} &= \tfrac{(n+1)\delta_3^2-2(n-1)\delta_3}{(n-1)^2},\\
\lambda_{333}=\lambda_{444} &= \tfrac{(n+1)\delta_3^2-2(n-1)\delta_3}{(n-1)^2},\\
\lambda_{341}=\lambda_{432} &= \tfrac{(n+1)\delta_1\delta_3^2-(n-1)\delta_1\delta_3-\varepsilon(n-1)\delta_3\sqrt{n\delta_1\delta_2}}{\delta_1(n-1)^2},\\
\lambda_{342}=\lambda_{431} &= \tfrac{(n+1)\delta_1\delta_3^2-(n-1)\delta_1\delta_3+\varepsilon(n-1)\delta_3\sqrt{n\delta_1\delta_2}}{\delta_1(n-1)^2}.
\end{aligned}
$$

2.4 Noncommutative Table Algebras of Dimension 5

Lemma 2.23. *Let (A, B) be a standard noncommutative reality-based algebra of dimension 5. Then (A, B) is a table algebra if and only if the following hold:*

$$(n+1)\delta_1 \geq 3(n-1) \tag{2.1}$$

$$(n+1)\delta_2 \geq 3(n-1) \tag{2.2}$$

$$(n+1)\delta_1\delta_2 \geq (n-1)\sqrt{n\delta_1\delta_2} \tag{2.3}$$

$$\delta_1[(n+1)\delta_3 - (n-1)] \geq (n-1)\sqrt{n\delta_1\delta_2} \tag{2.4}$$

$$\delta_2[(n+1)\delta_3 - (n-1)] \geq (n-1)\sqrt{n\delta_1\delta_2} \tag{2.5}$$

$$(n+1)\delta_3 \geq 2(n-1) \tag{2.6}$$

Proof.

($\Rightarrow$) We first assume that (A, B) is a table algebra; so, using the formulas for the structure constants, we can derive (2.1) - (2.6).

Step 1: $\lambda_{111} = \frac{(n+1)\delta_1^2 - 3(n-1)\delta_1}{(n-1)^2} \geq 0 \implies (n+1)\delta_1^2 - 3(n-1)\delta_1 = \delta_1((n+1)\delta_1 - 3(n-1)) \geq 0$. So, $\delta_1 \geq \frac{3(n-1)}{(n+1)}$ and (2.1) holds.

Similarly, $\lambda_{222} = \frac{(n+1)\delta_2^2 - 3(n-1)\delta_2}{(n-1)^2}$ gives us the condition $\delta_2 \geq \frac{3(n-1)}{(n+1)}$ and (2.2) holds.

Step 2: $\lambda_{123} = \lambda_{214} = \frac{(n+1)\delta_1\delta_2 + \varepsilon(n-1)\sqrt{n\delta_1\delta_2}}{(n-1)^2} \geq 0$

$$\begin{aligned}
(n+1)\delta_1\delta_2 + \varepsilon(n-1)\sqrt{n\delta_1\delta_2} &\geq 0 \\
(n+1)\delta_1\delta_2 &\geq -\varepsilon(n-1)\sqrt{n\delta_1\delta_2} \\
(n+1)^2(\delta_1\delta_2)^2 &\geq (n-1)^2 n\delta_1\delta_2 \\
(n+1)^2\delta_1\delta_2 &\geq (n-1)^2 n \\
\delta_1\delta_2 &\geq \tfrac{n(n-1)^2}{(n+1)^2}.
\end{aligned}$$

So, (2.3) holds.

Similarly, $\lambda_{124} = \lambda_{213} = \frac{(n+1)\delta_1\delta_2 - \varepsilon(n-1)\sqrt{n\delta_1\delta_2}}{(n-1)^2}$, $\lambda_{132} = \lambda_{412} = \frac{(n+1)\delta_1\delta_2\delta_3 + \varepsilon(n-1)\delta_3\sqrt{n\delta_1\delta_2}}{\delta_2(n-1)^2}$, $\lambda_{142} = \lambda_{312} = \frac{(n+1)\delta_1\delta_2\delta_3 - \varepsilon(n-1)\delta_3\sqrt{n\delta_1\delta_2}}{\delta_2(n-1)^2}$, $\lambda_{231} = \lambda_{421} = \frac{(n+1)\delta_1\delta_2\delta_3 - \varepsilon(n-1)\delta_3\sqrt{n\delta_1\delta_2}}{\delta_1(n-1)^2}$, and $\lambda_{241} = \lambda_{321} = \frac{(n+1)\delta_1\delta_2\delta_3 + \varepsilon(n-1)\delta_3\sqrt{n\delta_1\delta_2}}{\delta_1(n-1)^2}$ gives us the same condition.

Step 3: $\lambda_{133} = \lambda_{414} = \frac{(n+1)\delta_1\delta_3 - (n-1)\delta_1 - \varepsilon(n-1)\sqrt{n\delta_1\delta_2}}{(n-1)^2} \geq 0$.

$$\begin{aligned}
(n+1)\delta_1\delta_3 - (n-1)\delta_1 - \varepsilon(n-1)\sqrt{n\delta_1\delta_2} &\geq 0 \\
(n+1)\delta_1\delta_3 - (n-1)\delta_1 &\geq \varepsilon(n-1)\sqrt{n\delta_1\delta_2} \\
\delta_1((n+1)\delta_3 - (n-1)) &\geq \varepsilon(n-1)\sqrt{n\delta_1\delta_2} \\
\delta_1^2((n+1)\delta_3 - (n-1))^2 &\geq (n-1)^2 n\delta_1\delta_2 \\
\delta_1((n+1)\delta_3 - (n-1))^2 &\geq (n-1)^2 n\delta_2 \\
\delta_1 &\geq \tfrac{(n-1)^2 n\delta_2}{((n+1)\delta_3 - (n-1))^2}.
\end{aligned}$$

So, (2.4) holds.

Similarly, $\lambda_{144} = \lambda_{313} = \frac{(n+1)\delta_1\delta_3 - (n-1)\delta_1 + \varepsilon(n-1)\sqrt{n\delta_1\delta_2}}{(n-1)^2}$ and $\lambda_{341} = \lambda_{432} = \frac{(n+1)\delta_1\delta_3^2 - (n-1)\delta_1\delta_3 - \varepsilon(n-1)\delta_3\sqrt{n\delta_1\delta_2}}{\delta_1(n-1)^2}$ gives us the same condition.

Step 4: $\lambda_{233} = \lambda_{424} = \frac{(n+1)\delta_2\delta_3 - (n-1)\delta_2 + \varepsilon(n-1)\sqrt{n\delta_1\delta_2}}{(n-1)^2} \geq 0$.

$$\begin{aligned}
(n+1)\delta_2\delta_3 - (n-1)\delta_2 + \varepsilon(n-1)\sqrt{n\delta_1\delta_2} &\geq 0 \\
(n+1)\delta_2\delta_3 - (n-1)\delta_2 &\geq -\varepsilon(n-1)\sqrt{n\delta_1\delta_2} \\
\delta_2((n+1)\delta_3 - (n-1)) &\geq -\varepsilon(n-1)\sqrt{n\delta_1\delta_2} \\
\delta_2^2((n+1)\delta_3 - (n-1))^2 &\geq (n-1)^2 n\delta_1\delta_2 \\
\delta_2((n+1)\delta_3 - (n-1))^2 &\geq (n-1)^2 n\delta_1 \\
\delta_2 &\geq \tfrac{(n-1)^2 n\delta_1}{((n+1)\delta_3 - (n-1))^2}.
\end{aligned}$$

So, (2.5) holds.

Similarly, $\lambda_{244} = \lambda_{323} = \frac{(n+1)\delta_2\delta_3 - (n-1)\delta_2 - \varepsilon(n-1)\sqrt{n\delta_1\delta_2}}{(n-1)^2}$ and $\lambda_{342} = \lambda_{431} = \frac{(n+1)\delta_1\delta_3^2 - (n-1)\delta_1\delta_3 + \varepsilon(n-1)\delta_3\sqrt{n\delta_1\delta_2}}{\delta_1(n-1)^2}$ gives us the same condition.

Step 5: $\lambda_{343} = \lambda_{344} = \lambda_{433} = \lambda_{434} = \lambda_{333} = \lambda_{444} = \frac{(n+1)\delta_3^2 - 2(n-1)\delta_3}{(n-1)^2} \geq 0 \implies (n+1)\delta_3^2 - 2(n-1)\delta_3 \geq 0$. So, $\delta_3 \geq \frac{2(n-1)}{(n+1)}$ and (2.6) holds.

($\Leftarrow$) Now we assume that (2.1) - (2.6) hold; however, one can see from above, that it is easily verified that all the structure constants will be nonnegative. Thus, (A, B) is a table algebra. $\square$

Remark. The structure constants $\lambda_{112} = \lambda_{113} = \lambda_{114} = \frac{(n+1)\delta_1^2-(n-1)\delta_1}{(n-1)^2} \geq 0 \implies (n+1)\delta_1^2 - (n-1)\delta_1 = \delta_1((n+1)\delta_1 - (n-1))) \geq 0$. So, $\delta_1 \geq \frac{n-1}{n+1}$.

Similarly, $\lambda_{121} = \lambda_{211} = \frac{(n+1)\delta_1\delta_2-(n-1)\delta_2}{(n-1)^2}$ and $\lambda_{131} = \lambda_{141} = \lambda_{311} = \lambda_{411} = \frac{(n+1)\delta_1\delta_3-(n-1)\delta_3}{(n-1)^2}$ gives us the condition $\delta_1 \geq \frac{n-1}{n+1}$. However, we can ignore this condition since $\frac{3(n-1)}{(n+1)} > \frac{(n-1)}{(n+1)}$.

Also, $\lambda_{122} = \lambda_{212} = \frac{(n+1)\delta_1\delta_2-(n-1)\delta_1}{(n-1)^2}, \lambda_{221} = \lambda_{223} = \lambda_{224} = \frac{(n+1)\delta_2^2-(n-1)\delta_2}{(n-1)^2}$, and $\lambda_{232} = \lambda_{422} = \lambda_{242} = \lambda_{322} = \frac{(n+1)\delta_2\delta_3-(n-1)\delta_3}{(n-1)^2}$ gives us the condition $\delta_2 \geq \frac{n-1}{n+1}$. However, we can ignore this condition since $\frac{3(n-1)}{(n+1)} > \frac{(n-1)}{(n+1)}$.

Lemma 2.24. *For $a > \frac{1}{3}$, the polynomial $f(x) = x^3 + x^2 + ax - a$ is increasing in $(-\infty, \infty)$, and hence has a unique real root, denoted by $E(a)$.*

i. $0 < E(a) < 1$.

ii. $E(a) > E(b)$ *if* $a > b$.

iii. $E(a) \to 1$ *if* $a \to \infty$.

The root of $f(x)$ is given by:

$$E(a) = -\frac{1}{3} + \frac{3a-1}{3(-1+18a+3\sqrt{3}\sqrt{-a+11a^2+a^3})^{\frac{1}{3}}} + \frac{1}{3}(-1 + 18a + 3\sqrt{3}\sqrt{-a + 11a^2 + a^3})^{\frac{1}{3}}$$

Proof. Consider $f'(x) = 3x^2 + 2x + a$, which is an upward-facing parabola with vertex $(\frac{-1}{3}, \frac{-1}{3} + a)$. Since $a > \frac{1}{3}$, we have that $\frac{-1}{3} + a > 0$, which means that $f'(x) > 0$ always. So $f(x)$ is increasing on $(-\infty, \infty)$.

i. $f(0) = -a < 0$ and $f(1) = 2 > 0$. So, by the intermediate value theorem, there exists a real-valued root of $f(x)$ in $(0, 1)$. Hence $0 < E(a) < 1$. Also, since $f(x)$ is increasing on $(-\infty, \infty)$, $E(a)$ is the unique real root of $f(x)$.

ii. $E(a) = -\frac{1}{3} + \frac{3a-1}{3(-1+18a+3\sqrt{3}\sqrt{-a+11a^2+a^3})^{\frac{1}{3}}} + \frac{1}{3}(-1+18a+3\sqrt{3}\sqrt{-a + 11a^2 + a^3})^{\frac{1}{3}}$.
$E'(a) =$

$$\frac{9(-1+18a+3\sqrt{3}\sqrt{-a+11a^2+a^3})^{\frac{1}{3}}-(3a-1)\left(\frac{1}{(-1+18a+3\sqrt{3}\sqrt{-a+11a^2+a^3})^{\frac{2}{3}}}\right)\left(18+\frac{3\sqrt{3}(-1+22a+3a^2)}{2(-a+11a^2+a^3)^{\frac{1}{2}}}\right)}{9(-1+18a+3\sqrt{3}\sqrt{-a+11a^2+a^3})^2}$$

$+\left(\frac{1}{9(-1+18a+3\sqrt{3}\sqrt{-a+11a^2+a^3})^{\frac{2}{3}}}\right)\left(18+\frac{3\sqrt{3}(-1+22a+3a^2)}{2(-a+11a^2+a^3)^{\frac{1}{2}}}\right) > 0$ when $a > \frac{1}{3}$. So $E(a)$ is increasing, and so $E(a) > E(b)$ when $a > b$.

iii. Suppose that $\lim_{a\to\infty} E(a) = \beta < 1$. So, $E(a) < \beta < 1$ and $\beta^3+\beta^2+a\beta-a > 0$ for all $a > \frac{1}{3}$. Therefore

$$\begin{aligned} \beta^3+\beta^2 &> a - a\beta \\ &> a(1-\beta) \\ \tfrac{\beta^3+\beta^2}{1-\beta} &> a. \end{aligned}$$

So, $a < \frac{\beta^3+\beta^2}{1-\beta}$ for all $a > \frac{1}{3}$. Let $a = \frac{\beta^3+\beta^2}{1-\beta}+1$. But then we have $\frac{\beta^3+\beta^2}{1-\beta}+1 < \frac{\beta^3+\beta^2}{1-\beta}$ which yields $1 < 0$, a contradiction. Hence $\lim_{a\to\infty} E(a) = 1$.

□

Lemma 2.25. *Let (A, B) be a standard table algebra, then each $\delta_i \geq 1$. If each $\delta_i = 1$, then B is a group under multiplication.*

Proof. Assume that $B = \{b_0 = 1_A, b_1, ..., b_d\}$. Then for any $0 \leq i \leq d$,

$$b_i b_{i^*} = \sum_{j=0}^{d} \lambda_{ii^*j} b_j,\ \lambda_{ii^*0} = \delta_i,\ \lambda_{ii^*j} \geq 0.$$

Applying the degree map to both sides, we see that

$$\delta_i^2 = \sum_{j=0}^{d} \lambda_{ii^*j}\delta_j \geq \lambda_{ii^*0}\delta_0 = \delta_i,\ \delta_0 = 1,\ \lambda_{ii^*0} = \delta_i. \tag{2.7}$$

That is, $\delta_i^2 \geq \delta_i$, so $\delta_i \geq 1$.

Now assume that all $\delta_i = 1$, Then we prove that B is a group. From (2.7) we get that

$$1 = \sum_{j=0}^{d} \lambda_{ii^*j} = 1 + \sum_{j=1}^{d} \lambda_{ii^*j}.$$

So, $\lambda_{ii^*j} = 0$ for all $j \geq 1$, and $b_i b_{i^*} = b_0 = 1_A$. That is, b_i is invertible, and $b_i^{-1} = b_{i^*}$. Then

$$b_j = b_0 b_j = (b_{i^*} b_i) b_j = \sum_{m=0}^{d} \lambda_{ijm} b_{i^*} b_m.$$

Since the structure constants are non-negative, the terms appearing in the right hand cannot cancel each other. Thus for any m such that $\lambda_{ijm} \neq 0$, $b_{i^*} b_m = \alpha_m b_j$ for some α_m. Applying the degree map, we get that $b_{i^*} b_m = b_j$ for all m such that $\lambda_{ijm} \neq 0$. Thus if there are at least two distinct m, say m_1 and m_2, such that $\lambda_{ijm_1} \neq 0$, $\lambda_{ijm_2} \neq 0$, then $b_{i^*} b_{m_1} = b_{i^*} b_{m_2}$. Hence $b_i(b_{i^*} b_{m_1}) = b_i(b_{i^*} b_{m_2})$, and $b_{m_1} = b_{m_2}$, a contradiction. This proves that there is only one m such that $\lambda_{ijm} \neq 0$, and $b_i b_j = b_m$. So B is a group by the definition of a group. □

Lemma 2.26. *Let (A, B) be a noncommutative standard table algebra of dimension 5, then $n > 5$.*

Proof. If $n = 5$, then each $\delta_i = 1$. Hence B is a group (under multiplication) of order 5. Hence B is abelian, and (A, B) is commutative, a contradiction. □

Lemma 2.27. $f(x) = \frac{(x-1)(1+\sqrt{x})}{x+1}$ *for $x > 5$ is an increasing function.*

Proof. Consider $f'(x)$.

$$\begin{aligned} f'(x) &= \tfrac{(x+1)(1+\sqrt{x}+(x-1)\frac{1}{2\sqrt{x}})-(x-1)(1+\sqrt{x})}{(x+1)^2} \\ &= \tfrac{(x+1)(2\sqrt{x}+2x+x-1)-(x-1)(2\sqrt{x}+2x)}{2\sqrt{x}(x+1)^2} \\ &= \tfrac{2x\sqrt{x}+2\sqrt{x}+3x^2+3x-x-1-(2x\sqrt{x}-2\sqrt{x}+2x^2-2x)}{2\sqrt{x}(x+1)^2} \\ &= \tfrac{x^2+4x+4\sqrt{x}-1}{2\sqrt{x}(x+1)^2}. \end{aligned}$$

Since $x > 5$, $x^2 + 4x + 4\sqrt{x} - 1 > 0$ and $2\sqrt{x}(x+1)^2 > 0$. So, $f'(x) > 0$ and $f(x)$ is increasing. □

Remark. From (2.4) and (2.5), $(n+1)\delta_3 - (n-1) \geq (n-1)\sqrt{n}\sqrt{\frac{\delta_2}{\delta_1}}$ and $(n+1)\delta_3 - (n-1) \geq (n-1)\sqrt{n}\sqrt{\frac{\delta_1}{\delta_2}}$. Since either $\frac{\delta_1}{\delta_2} \geq 1$ or $\frac{\delta_2}{\delta_1} \geq 1$, we always have $(n+1)\delta_3 - (n-1) \geq (n-1)\sqrt{n}$. So $(n+1)\delta_3 \geq (n-1)(1+\sqrt{n})$. Now it follows from $n > 5$ that (2.6) is a direct consequence of (2.4) and (2.5).

Theorem 2.28. *If (A, B) is a table algebra, then*

$$\delta_1 > 3, \delta_2 > 3, \delta_3 \geq 2 + \frac{6\sqrt{5}}{5},$$

and

$$9 + 4\sqrt{5} \leq n \leq \frac{2}{1 - E(\delta_1\delta_2)} - 1,$$

where $E(\delta_1\delta_2)$ is the (unique) real root of $x^3 + x^2 + \delta_1\delta_2 x - \delta_1\delta_2 = 0$.

Proof. Take the square of both sides of (2.3) and we get

$$\left(\frac{n-1}{n+1}\right)^2 n \leq \delta_1\delta_2 \tag{2.8}$$

Thus, $\delta_1 + \delta_2 \geq 2\sqrt{\delta_1\delta_2} \geq 2\sqrt{n}\frac{n-1}{n+1}$. From (2.4) and (2.5) we have

$$(n+1)\delta_3 - (n-1) \geq (n-1)\sqrt{n} \cdot \max\left\{\sqrt{\frac{\delta_2}{\delta_1}}, \sqrt{\frac{\delta_1}{\delta_2}}\right\}.$$

Since either $\frac{\delta_2}{\delta_1} \geq 1$ or $\frac{\delta_1}{\delta_2} \geq 1$, we always have $(n+1)\delta_3 - (n-1) \geq (n-1)\sqrt{n}$. Thus

$$\delta_3 \geq \frac{n-1}{n+1}(1 + \sqrt{n}). \tag{2.9}$$

Therefore,

$$n = 1 + \delta_1 + \delta_2 + 2\delta_3 \geq 1 + 2\sqrt{n}\left(\frac{n-1}{n+1}\right) + 2(1 + \sqrt{n})\left(\frac{n-1}{n+1}\right).$$

Hence, $\left(\frac{n-1}{n+1}\right)(1 + 4\sqrt{n} - n) \leq 0$. Since $n > 5$, we see that $1 + 4\sqrt{n} - n \leq 0$. So $n \geq 9 + 4\sqrt{5}$.

Now from (2.1), $\delta_1 \geq \frac{3(n-1)}{(n+1)} \geq \frac{3(8+4\sqrt{5})}{10+4\sqrt{5}} = \frac{6\sqrt{5}}{5}$. Similarly from (2.2) we see that $\delta_2 \geq \frac{6\sqrt{5}}{5}$. Furthermore, (2.9) implies that

$$\delta_3 \geq \frac{n-1}{n+1}(1 + \sqrt{n}) \geq \frac{8 + 4\sqrt{5}}{10 + 4\sqrt{5}}(1 + \sqrt{9 + 4\sqrt{5}}) = 2 + \frac{6\sqrt{5}}{5}.$$

Let $x = 1 - \frac{2}{n+1}$. Since $n \geq 9 + 4\sqrt{5}$, we have $x > 0, 1 - x > 0$, and (2.8) is equivalent to

$$x^3 + x^2 + \delta_1\delta_2 x - \delta_1\delta_2 \leq 0. \tag{2.10}$$

Note that $\delta_1\delta_2 \geq \frac{4(9+\sqrt{5})}{5}$ by (2.8). So (2.10) holds if and only if $x \leq E(\delta_1\delta_2)$. Thus, (2.8) holds if and only if $n \leq \frac{2}{1-E(\delta_1\delta_2)} - 1$.

Towards a contradiction, assume that $\delta_1 \leq 3$. Then from (2.8) $\delta_2 \geq \frac{\left(\frac{n-1}{n+1}\right)^2 n}{\delta_1} \geq \frac{\left(\frac{n-1}{n+1}\right)^2 n}{3}$. Also $\frac{\delta_2}{\delta_1} \geq \frac{\delta_2}{3} \geq \frac{\left(\frac{n-1}{n+1}\right)^2 n}{9}$. Hence from (2.4),

$$\begin{aligned}(n+1)\delta_3 - (n-1) &\geq (n-1)\sqrt{n}\sqrt{\frac{\delta_2}{\delta_1}} \\ &\geq (n-1)\sqrt{n}\left[\frac{\left(\frac{n-1}{n+1}\right)n}{3}\right] \\ &= \frac{(n-1)^2}{(n+1)}\left(\frac{n}{3}\right).\end{aligned}$$

Thus, $\delta_3 \geq \frac{n-1}{n+1}\left[1 + \frac{\left(\frac{n-1}{n+1}\right)n}{3}\right]$. We have already shown that $\delta_1 \geq \frac{6\sqrt{5}}{5} > 2$. Therefore, $n = 1 + \delta_1 + \delta_2 + 2\delta_3 \geq 3 + \frac{\left(\frac{n-1}{n+1}\right)^2 n}{3} + 2\left(\frac{n-1}{n+1}\right)\left[1 + \frac{\left(\frac{n-1}{n+1}\right)n}{3}\right] = 3 + 2\left(\frac{n-1}{n+1}\right) + \left(\frac{n-1}{n+1}\right)^2 n$. So $n(n+1)^2 \geq 3(n+1)^2 + 2(n-1)(n+1) + (n-1)^2 n$. Thus, $n^2 + 6n + 1 \leq 0$, a contradiction to $n \geq 9 + 4\sqrt{5}$. This proves that $\delta_1 > 3$. Similarly, we can prove that $\delta_2 > 3$.

□

Corollary 2.29. *For any $n \geq 9 + 4\sqrt{5}$, there is a standard noncommutative table algebra of dimension 5 such that*

$$\delta_1 = \delta_2 = \frac{(n-1)}{n+1}\sqrt{n}, \delta_3 = \frac{(n-1)(n+1-2\sqrt{n})}{2(n+1)}.$$

Proof. Since $n \geq 9 + 4\sqrt{5}$, $\sqrt{n} \geq \sqrt{9 + 4\sqrt{5}} \approx 4.236... > 3$. So, $\frac{(n-1)}{n+1}\sqrt{n} > \frac{3(n-1)}{n+1}$, which satisfies (2.1) and (2.2).
$\delta_1 \cdot \delta_2 = \left(\frac{(n-1)}{n+1}\sqrt{n}\right)^2 = \frac{(n-1)^2}{(n+1)^2}n$, which satisfies (2.3).
Substituting in the given $\delta_1, \delta_2, \delta_3$ into (2.4) and (2.5) gives us the inequality $(n-1) \geq \frac{4n}{(n-1-2\sqrt{n})^2}$. Substituting in $n = 9 + 4\sqrt{5}$, we get $16.944... \geq \frac{71.777...}{71.775...} \approx 1$. So (2.4) and (2.5) is satisfied.
Also, $\frac{n+1-2\sqrt{n}}{2} \geq 5 + 2\sqrt{5} - \sqrt{9+4\sqrt{5}} \approx 5.236... > 2$. Thus, $\frac{(n-1)(n+1-2\sqrt{n})}{2(n+1)} > \frac{2(n-1)}{(n+1)}$, which satisfies (2.6). □

Example 2.30. Let $n = 9+4\sqrt{5}, \delta_1 = \delta_2 = \frac{2(5+2\sqrt{5})}{5}$, and $\delta_3 = 2+\frac{6\sqrt{5}}{5}$. Then from Corollary 2.1, this forms a standard noncommutative table algebra of dimension 5.

Corollary 2.31. *For any $n \geq 9+4\sqrt{5}$, there is a standard noncommutative table algebra of dimension 5 such that*

$$\delta_1 = \delta_2 = \frac{(n-1)(n-1-2\sqrt{n})}{2(n+1)}, \delta_3 = \frac{n-1}{n+1}(1+\sqrt{n}).$$

Proof. Since $n \geq 9+4\sqrt{5}$, $\frac{n-1-2\sqrt{n}}{2} \geq 4+2\sqrt{5}-\sqrt{9+4\sqrt{5}} \approx 4.236... > 2$. Thus, $\frac{(n-1)(n-1-2\sqrt{n})}{2(n+1)} > \frac{3(n-1)}{(n+1)}$, which satisfies (2.1) and (2.2).
$\delta_1 \cdot \delta_2 = \left(\frac{(n-1)(n-1-2\sqrt{n})}{2(n+1)}\right)^2 = \frac{(n-1)^2(n-1-2\sqrt{n})^2}{4(n+1)^2}$. $\frac{(n-1-2\sqrt{n})^2}{4} \geq \frac{(8+4\sqrt{5}-\sqrt{9+4\sqrt{5}})^2}{4} \approx 40.373... > 9+4\sqrt{5}$, which satisfies (2.3).
Substituting in the given $\delta_1, \delta_2, \delta_3$ into (2.4) and (2.5) gives us the inequality $\delta_1, \delta_2 \geq \frac{(n-1)(n-1-2\sqrt{n})}{2(n+1)}$ but actually $\delta_1, \delta_2 = \frac{(n-1)(n-1-2\sqrt{n})}{2(n+1)}$ so (2.4) and (2.5) are satisfied.
Also, $1+\sqrt{n} \geq 1+\sqrt{9+4\sqrt{5}} \approx 5.236... > 2$, which satisfies (2.6). □

Theorem 2.32. *Let (A, B) be a standard noncommutative reality-based algebra of dimension 5.*

i. For an $\alpha > 3$, there is a noncommutative table algebra of dimension 5 such that $\delta_1 = \alpha$ (or $\delta_2 = \alpha$).

ii. If $n \geq 9+4\sqrt{5}, \delta_1 \geq 3, \delta_2 \geq 3, E(\delta_1\delta_2) \geq 1-\frac{2}{n+1}$, and $\delta_3 \geq \max\{\delta_1, \delta_2\}+\frac{n-1}{n+1}$, then (A, B) is a table algebra.

Proof.

i. Since $\alpha > 3$, there is a positive real number β such that for all $n \geq \beta$,

$$(\alpha-3)n^3 + (6-\alpha-\alpha^2)n^2 - (\alpha+2\alpha^2+3)n - \alpha(\alpha-1) \geq 0 \qquad (2.11)$$

and

$$n^3 - (2+3\alpha)n^2 + (1-6\alpha)n - 3\alpha > 0. \qquad (2.12)$$

Let $\delta_1 = \alpha, \delta_2 = \frac{\left(\frac{n-1}{n+1}\right)^2 n}{\alpha}$, and $\delta_3 = \frac{1}{2}(n-1-\delta_1-\delta_2)$. Then $\delta_2 > 3$ by (2.12), and by choosing n large enough, we may assume that $\delta_2 \geq \delta_1$. So by (2.11) $\delta_3 \geq \frac{n-1}{n+1}\left(1+\sqrt{n\frac{\delta_2}{\delta_1}}\right)$. Hence the reality-based algebra is a table algebra. The proof of $ii.$ is straightforward and follows directly from Theorem 2.28.

$\square$

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