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Monochromatic Columns: The Prime Case

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DEDICATION

I dedicate this thesis to my family and church family for their love and support. I also dedicate this thesis to Steve Goggin who ignited my passion for mathematics and education.

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I would like to thank my advisory committee, especially Dr. Steve Szabo who introduced me to the problem. I thank him for his time, patience, and encouragement throughout this process. I thank him for challenging me to understand mathematics at a higher level and helping me accomplish a task that I did not know I was capable of.

ABSTRACT

Let $p_1, p_2, \ldots, p_n$ be pairwise coprime positive integers and let $P = p_1 p_2 \cdots p_n$. Let $0, 1, \ldots, m-1$ be a sequence of m different colors. Let A be an $n \times mP$ matrix of colors in which row i consists of blocks of p_i consecutive entries of the same color, with colors 0 through $m-1$ repeated cyclically. The Monochromatic Column problem is to determine the number of columns of A in which every entry is the same color. A partial solution for the case when m is prime is given.

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Chapter 1

Introduction

Let m be a number of colors where each color is represented by one of the integers in $\{0, 1, \ldots, m-1\}$. Let $p_1, \ldots, p_n$ be positive integers and $P = p_1 p_2 \cdots p_n$. These colors are arranged in an $n \times mP$ matrix where row i is comprised of cyclic successions of p_i blocks of the same color. The Monochromatic Column Problem is to count the columns where the entries in the column are the same.

The Monochromatic Column Problem was initiated by Nagpaul and Jain after distilling down a problem from molecular and structural biology. This problem is discussed in [2], where Jiang, Kearney, and Li look for an optimal score for an alignment of sequences of characters from a fixed alphabet. In [1], Jain and Nagpaul, solved the Monochromatic Column Problem for two colors and in [4], Szabo and Srivastava solved the problem for three colors. In this work, we will give a partial solution to the Monochromatic Column Problem for a prime number of colors. In the works mentioned, as well as ours, it is assumed that $p_1, \ldots, p_n$ were pairwise coprime. Furthermore, the main tool used in the solution, is the Chinese Remainder Theorem.

In this paper, we will give a solution to three specific cases of the Monochromatic Column Problem for a prime number of colors. First, we consider the case when $p_1, \ldots, p_n$ are pairwise coprime integers and congruent to a fixed $s \in \{1, \ldots, q-1\}$. Then, we consider the case when $p_1, \ldots, p_n$ are pairwise coprime integers and are congruent to either b_1 or b_2. Finally, we consider the case when we assume that we already know the number of monochromatic columns in some matrix B associated with pairwise coprime integers $p_1 \ldots, p_{n-1}$. This case counts the number of monochromatic columns in a matrix A which is associated with pairwise coprime integers, $p_1 \ldots, p_n$, where $q \mid p_n$.

Our solution is a generalization of the solution given in the three color problem. It encapsulates the solutions for the two and three color problems which were previously given. When the number of colors is a prime larger than three, what we have only provides a partial solution. Example 3.1 is actually a counterexample showing that the techniques used so far in solving various aspects of the Monochromatic Column Problem cannot be used to find a full solution to the problem with a prime number of colors.

Chapter 2

Preliminaries

Let m be a positive integer. The colors for m are represented by the integers $0, 1, \ldots, m-1$. An $n \times s$ *color matrix* is an $n \times s$ matrix $A = (a_{ij})$ in which every entry is one of the m colors. Column j of A is a *monochromatic column* if there exists a $c \in \{0, \ldots, m-1\}$ such that, for $1 \leq i \leq n, a_{ij} = c$. Column j of A is a *bichromatic column* if there exist $c_u, c_l \in \{0, \ldots, m-1\}$ such that, for $1 \leq i \leq h, a_{ij} = c_u$, and, $h+1 \leq i \leq n, a_{ij} = c_l$. If $c_u = c_l$, then j is a monochromatic column. For a positive integer d, row i of A is d-blocked with initial color ρ if $d \mid s$ and, for $1 \leq j \leq s$, the i, j-entry is the color

$$a_{ij} = \left(\left\lfloor \frac{j-1}{d} + \rho \right\rfloor \right) \bmod m.$$

The matrix A is the $(d_1, d_2, \ldots, d_n; \rho_1, \rho_2, \ldots, \rho_n)$ *color matrix of width* s if, for every i satisfying $1 \leq i \leq n$, we have $d_i \mid s$ and row i of A is d_i-blocked with initial color ρ_i. Let $P = p_1 p_2 \cdots p_n$. Note that the permutation of rows will not affect the number of monochromatic columns. The notation $(m; p_1, p_2, \ldots, p_n; \rho_1, \rho_2, \ldots, \rho_r; t_1, \ldots, t_{r-1})$ will denote a color matrix such that row i is p_i-blocked and rows $t_{k-1}+1$ to t_k start with color ρ_k for $1 \leq k \leq r$, defining $t_0 = 0$ and $t_r = n$. For the case when $r = 1$, the color matrix will be denoted by $(m; p_1, \ldots, p_n; \rho_1)$. The *Monochromatic Column Problem* is to determine the count $N(m; p_1, p_2, \ldots, p_n; \rho_1, \rho_2, \ldots, \rho_r; t_1, \ldots, t_{r-1})$, of monochromatic columns in the $(m; p_1, p_2, \ldots, p_n; \rho_1, \rho_2, \ldots, \rho_r; t_1, \ldots, t_{r-1})$ color matrix A.

To illustrate the layout of a color matrix, let $m = 5, n = 2, p_1 = 2$, and $p_2 = 3$. The color matrix $(5; 2, 3; 0)$ is

$$A = \begin{pmatrix} 0&0&1&1&2&2&3&3&4&4&0&0&1&1&2&2&3&3&4&4&0&0&1&1&2&2&3&3&4&4 \\ 0&0&0&1&1&1&2&2&2&3&3&3&4&4&4&0&0&0&1&1&1&2&2&2&3&3&3&4&4&4 \end{pmatrix}.$$

For $i = 1, 2, ..., n$ and $j = 1, 2, ..., mP$, define

$$b_{ij} = \left\lfloor \frac{j-1}{p_i} \right\rfloor$$

and

$$k_{ij} = j - b_{ij}p_i.$$

Then, $1 \leq k_{ij} \leq p_i$, and the color in entry i, j is

$$a_{ij} = (b_{ij} + \rho_i) \bmod m,$$

where ρ_i is the initial color of row i.

The following well-known results will be needed in the proof of the Monochromatic Column Problem.

Lemma 2.1.

1. *For $a, b \in \mathbb{Z}$, there are $b - a + 1$ integer solutions to $a \leq x \leq b$.*
2. $-\lceil a \rceil = \lfloor -a \rfloor$

Lemma 2.2 (Fermat's Little Theorem)**.** *Let q denote a prime. If $q \nmid r$ then $r^{q-1} \equiv 1(\text{mod} q)$. For every integer $r, r^q \equiv r(\bmod q)$.*

Lemma 2.3 (Theorem 2.6, [3])**.** *If $gcd(r, q) = 1$, then there is an x such that $rx \equiv 1(\bmod q)$. Any two such x are congruent modulo q. If $gcd(r, q) > 1$, then there is no such x.*

Proof. Let $gcd(r, q) = h$. If $rx \equiv 1(\text{mod} q)$, then $rx \equiv 1(\text{mod} h)$, and, since $h \times r$, we have $0x \equiv 1(\bmod h)$, which is a contradiction unless $h = 1$. Let $h = 1$. To show uniqueness, if $rx \equiv 1(\text{mod} q)$ and $ry \equiv 1(\bmod q)$, then $rx \equiv ry(\bmod q)$.

Since $gcd(r, q) = 1, x \equiv y (\bmod q)$. To show existence, since $gcd(r, q) = 1$, there exist integers x, y such that $rx + qy = 1$. When we take this equation modulo q, we get $rx \equiv 1 (\bmod q)$, and so $x = r^{-1}$, where r^{-1} is the multiplicative inverse of r which we know exists since $gcd(r, q) = 1$. □

Lemma 2.4 (Chinese Remainder Theorem)**.** *Let $p_1, p_2, \ldots, p_n$ be pairwise coprime integers and let $a_1, a_2, \ldots a_n$ be any n integers. Then the congruences*

$$\begin{aligned} x &\equiv a_1 (\bmod p_1) \\ x &\equiv a_2 (\bmod p_2) \\ &\vdots \\ x &\equiv a_r (\bmod p_n) \end{aligned}$$

have common solutions. If x_0 is one such solution, then an integer x satisfies the congruences if and only if x is of the form $x = x_0 + kp_1p_2 \cdots p_n$ for some integer k.

Proof. Let $P = p_1p_2 \cdots p_r$. Since $p_1, \ldots, p_n$ are pairwise coprime integers, and let $(\frac{P}{p_i}, p_i) = 1$. By Lemma 2.2, we know that if $gcd(\frac{P}{p_i}, p_i) = 1$, then there is some b_i such that $(\frac{P}{p_i})b_i \equiv 1 (\bmod p_i)$, and any two such b_i are congruent modulo P. Hence, for each i, there is an integer b_i such that $(\frac{P}{p_i})b_i \equiv 1 (\bmod p_i)$. Clearly, $(\frac{P}{p_i})b_i \equiv 0 (\bmod p_j)$ if $i \neq j$ because $(\frac{P}{p_i})b_i$ would be divisible by p_j. Let $x = \sum_{i=1}^{n} \frac{P}{p_i} b_i a_i$, then we find that $x \equiv \frac{P}{p_j} b_j a_j \equiv a_j (\bmod p_j)$. Thus x is a solution of the system. If x and x_1 are two solutions of the system, then $x \equiv x_1 (\bmod p_i)$ for $i = 1, 2, \cdots, n$, and hence $x \equiv x_1 (\bmod P)$. □

Example 2.1. We use the Chinese Remainder Theorem to find a solution to the system of linear congruences.

$$\begin{aligned} x &\equiv 1 (\bmod 5) \\ x &\equiv 2 (\bmod 6) \\ x &\equiv 3 (\bmod 7). \end{aligned}$$

The moduli are relatively prime, as required by the CRT. Let $p_1 = 5, p_2 = 6, p_3 = 7$ and $P = 5 \cdot 6 \cdot 7 = 210$. Let a_i be the remainders

$$a_1 = 1$$
$$a_2 = 2$$
$$a_3 = 3.$$

Next, we want to find some b_i such that $p_i b_i \equiv 1 (\bmod p_i)$. So, we solve for b_1, b_2, and b_3, obtaining

$$\begin{aligned} \frac{P}{p_1} b_1 &\equiv 1 (\bmod 5) \\ 42 b_1 &\equiv 1 (\bmod 5) \\ 2 b_1 &\equiv 1 (\bmod 5) \\ b_1 &= 3 \end{aligned}$$

$$\begin{aligned} \frac{P}{p_2} b_2 &\equiv 1 (\bmod 6) \\ 35 b_2 &\equiv 1 (\bmod 6) \\ 5 b_2 &\equiv 1 (\bmod 6) \\ b_2 &= 5 \end{aligned}$$

$$\begin{aligned} \frac{P}{p_3} b_3 &\equiv 1 (\bmod 7) \\ 30 b_3 &\equiv 1 (\bmod 7) \\ 2 b_3 &\equiv 1 (\bmod 7) \\ b_3 &= 4. \end{aligned}$$

Next, we generate the solution to the system by evaluating the summation that was constructed in the proof of the CRT. So, let $x = \sum_{i=1}^{3} \frac{P}{p_i} b_i a_i = 836$. Then,

$$\begin{aligned} x &\equiv 836 (\bmod 210) \\ &\equiv 206 (\bmod 210). \end{aligned}$$

Therefore, the solution to the system will be any number in the form $206 + 210n$, where n is an integer.

Chapter 3

The Monochromatic Column Problem

The Monochromatic Problem for some prime number of colors, q, will be partially solved. In the color matrix $(q; p_1, \ldots, p_n; 0, 1, \ldots, q-1; t_1, \ldots, t_{q-1})$, we want to count the number of monochromatic columns. The first proposition will solve the case where our p_i are congruent modulo q. The second proposition will solve the case where, given $b_1 \neq b_2$, all p_i are congruent to either b_1 modulo q or b_2 modulo q. Lastly, the third proposition will solve the case where we already know the number of monochromatic columns for a color matrix associated to a set of positive pairwise coprime integers $p_1, \ldots, p_{n-1}$ of which none are divisible by q and we want to find the number of monochromatic columns of a matrix associated to positive pairwise coprime integers $p_1, \ldots, p_n$ where $p_n | q$. After the third proposition, a counterexample is presented that shows that there is no general solution using the same method that was used in the propositions.

Proposition 3.1. *Let q be a prime number, $s \in \{1, ..., q-1\}$, and n be a positive integer. Let $p_1, ..., p_n$ be positive, pairwise coprime integers such that $p_i \equiv s (\bmod q)$, $t_0, ..., t_q$ be integers that satisfy $0 = t_0 \leq \cdots \leq t_q = n$, and $N = N(q; p_1, p_2, ..., p_n; 0, 1, ..., q-1; t_1, ..., t_{q-1})$. Then*

$$N = q \sum_{\beta=0}^{q-1} \prod_{\rho=0}^{q-1} \prod_{i=t_p+1}^{t_{p+1}} \frac{p_i - s}{q} + \left\lfloor \frac{s - (\beta + \rho_i s) \bmod q}{q} \right\rfloor + \left\lfloor \frac{(\beta + \rho_i s) \bmod q - 1}{q} \right\rfloor + 1.$$

Proof. Let $P = p_1 p_2 \cdots p_n$. Let A be the $(q; p_1, p_2, ..., p_n; 0, 1, ..., q-1; t_1, ..., t_{q-1})$ color matrix. Denote the initial color of row i by ρ_i. By definition,

$$a_{ij} = \left(\left\lfloor \frac{j-1}{p_i} \right\rfloor + \rho_i \right) \bmod q.$$

This calculates the color of the element because $\left\lfloor \frac{j-1}{p_i} \right\rfloor$ calculates the number of complete blocks that the element is away from the beginning of the row. The number of blocks is added to the starting color of the row, ρ_i, and then taking this modulo q gives the color. We will first show that if column j is monochromatic, then a column some multiple of P away is also monochromatic.

Let $1 \le i_1 \le i_2 \le n$, $1 \le j \le P$, and $0 \le \alpha \le q-1$. Since $p_{i_1} \equiv p_{i_2} (\bmod q)$,

$$\begin{aligned}
(a_{i_1,j} - a_{i_1,\alpha P+j}) &= \left(\left\lfloor \frac{j-1}{p_{i_1}} \right\rfloor + \rho_{i_1} - \left\lfloor \frac{\alpha P + j - 1}{p_{i_1}} \right\rfloor - \rho_{i_1} \right) \\
&= \left(\left\lfloor \frac{j-1}{p_{i_1}} \right\rfloor - \frac{\alpha P}{p_{i_1}} - \left\lfloor \frac{j-1}{p_{i_1}} \right\rfloor \right) \\
&= \left(\left\lfloor \frac{j-1}{p_{i_2}} \right\rfloor - \frac{\alpha P}{p_{i_2}} - \left\lfloor \frac{j-1}{p_{i_2}} \right\rfloor \right) \\
&= \left(\left\lfloor \frac{j-1}{p_{i_2}} \right\rfloor + \rho_{i_2} - \left\lfloor \frac{\alpha P + j - 1}{p_{i_2}} \right\rfloor - \rho_{i_2} \right) \\
&= (a_{i_2,j} - a_{i_2,\alpha P+j}) \bmod q.
\end{aligned}$$

This shows that if column j is monochromatic, then so is column $\alpha P + j$. Hence, it suffices to count the number of monochromatic columns in the first P columns of A and multiply by q.

Since $p_1, \ldots, p_n$ are pairwise coprime integers, the Chinese Remainder Theorem guarantees a unique $j \in \{1, 2, \ldots, P\}$ such that $j \equiv v_i \bmod p_i$ for each $i \in \{1, 2, \ldots, n\}$. So, by the CRT, there is a one-to-one correspondence between an n-tuple, $(v_1, v_2, \ldots, v_n)$, such that $1 \le v_i \le p_i$ and the set $\{1, 2, \ldots, P\}$. Thus, the mapping of a column $j \in \{1, 2, \ldots, P\}$ to $(k_{1j}, k_{2j}, \ldots, k_{nj})$ is a one-to-one correspondence. If we can identify the n-tuples that map to a monochromatic column, then we can count the number of monochromic columns.

Now, for $1 \leq i \leq n$ and $1 \leq j \leq P$, since $k_{ij} = j - b_{ij}p_i = j - \lfloor (j-1)/p_i \rfloor p_i$, the color in entry i, j is

$$a_{ij} = \left(\left\lfloor \frac{j-1}{p_i} \right\rfloor + \rho_i \right) \bmod q$$

$$a_{ij} = \left(\frac{j - k_{ij}}{p_i} + \rho_i \right) \bmod q.$$

So, $k_{ij} \equiv j - a_{ij}p_i + \rho_i p_i (\bmod\, q)$. Since $p_i \equiv s (\bmod\, q)$, we have that $k_{ij} \equiv j - a_{ij}s + \rho_i s (\bmod\, q)$.

Let $j \in \{1, 2, \ldots, P\}$, and define $\beta = (j - a_{1j}s) \bmod q$. Then, column j is monochromatic if and only if $k_{ij} \equiv \beta + \rho_i s (\bmod\, q)$ for all $i \in \{1, 2, ..., n\}$. So, setting $x_i = \frac{(k_{ij} - ((\beta + \rho_i s) \bmod q)}{q}$, we have that $1 \leq qx_i + (\beta + \rho_i s) \bmod q \leq p_i$ or, equivalently, $\frac{1 - (\beta + \rho_i s) \bmod q}{q} \leq x_i \leq \frac{p_i - (\beta + \rho_i s) \bmod q}{q} = \frac{p_i - s}{q} - \frac{(\beta + \rho_i s) \bmod q - s}{q}$, where x_i is an integer. Now, by Lemma 2.1, the number of integer solutions to

$$\left\lceil \frac{1 - (\beta + \rho_i s) \bmod q}{q} \right\rceil \leq x \leq \left\lfloor \frac{p_i - s}{q} - \frac{(\beta + \rho_i s) \bmod q - s}{q} \right\rfloor$$

is

$$\begin{aligned} & \left\lfloor \frac{p_i - s}{q} - \frac{(\beta + \rho_i s) \bmod q - s}{q} \right\rfloor - \left\lceil \frac{1 - (\beta + \rho_i s) \bmod q}{q} \right\rceil + 1 \\ = & \frac{p_i - s}{q} + \left\lfloor \frac{s - (\beta + \rho_i s) \bmod q}{q} \right\rfloor + \left\lfloor \frac{(\beta + \rho_i s) \bmod q - 1}{q} \right\rfloor + 1. \end{aligned}$$

Summing over the q possibilities for β, multiplying the number of solutions for each row, and multiplying the sum by q, we find that the number of monochromatic columns in A is

$$N = q \sum_{\beta=0}^{q-1} \prod_{\rho=0}^{q-1} \prod_{i=t_p+1}^{t_{p+1}} \frac{p_i - s}{q} + \left\lfloor \frac{s - (\beta + \rho s) \bmod q}{q} \right\rfloor + \left\lfloor \frac{(\beta + \rho s) \bmod q - 1}{q} \right\rfloor + 1.$$

□

Remark 3.1. Note that the expression $\left\lfloor \frac{s-(\beta+\rho_i s) \bmod q}{q} \right\rfloor + \left\lfloor \frac{(\beta+\rho_i s) \bmod q-1}{q} \right\rfloor$ in the previous proposition is either -1 or 0.

Proposition 3.2. *Let q be a prime number, $b_1, b_2 \in \{1, \ldots, q-1\}$ such that $b_1 \neq b_2$, and n and h be positive integers such that $1 \leq h < n$. Let $p_1, \ldots, p_n$ be positive, pairwise coprime integers such that $p_i \equiv b_1 (\operatorname{mod} q)$ for $1 \leq i \leq h$ and $p_i \equiv b_2 (\operatorname{mod} q)$ for $h+1 \leq i \leq n$, and $t_0, \ldots, t_{2q}$ be integers that satisfy $0 = t_0 \leq \cdots \leq t_q = h \leq t_{q+1} \leq \cdots \leq t_{2q} = n$. Let $N = N(q; p_1, \ldots, p_n; 0, \ldots, q-1, 0, \ldots, q-1; t_1, \ldots, t_{2q-1})$. Then*

$$N = \sum_{\beta_1=0}^{q-1} \sum_{\beta_2=0}^{q-1} \prod_{s=1}^{2} \prod_{\rho=0}^{q-1} \prod_{i=t_{q(s-1)+\rho}+1}^{i=t_{q(s-1)+\rho+1}} \left(\frac{p_i - b_s}{q} + \left\lfloor \frac{b_s-(\beta_s+\rho b_s) \bmod q}{q} \right\rfloor + \left\lfloor \frac{(\beta_s+\rho b_s) \bmod q+q-1}{q} \right\rfloor \right).$$

Proof. Let $P = p_1 \cdots p_n$. Let A be the $(q; p_1, p_2, \ldots, p_n; 0, \ldots, q-1, 0, \ldots, q-1; t_1, \ldots, t_{2q-1})$ color matrix. Denote the initial color of row i by ρ_i. By definition,

$$a_{ij} = \left(\left\lfloor \frac{j-1}{p_i} \right\rfloor + \rho_i \right) \bmod q.$$

This calculates the color of the element because $\left\lfloor \frac{j-1}{p_i} \right\rfloor$ calculates the number of complete blocks that the element is away from the beginning of the row. The number of blocks is added to the starting color of the row, ρ_i, and then taking this modulo q gives the color. We will show that if column j is h-bichromatic, then a column some multiple of P away is also monochromatic..

Let $1 \leq i_1 \leq i_2 \leq h, 1 \leq j \leq P$, and $0 \leq \alpha \leq q-1$. Since $p_{i_1} \equiv p_{i_2} (\operatorname{mod} q)$,

$$\begin{aligned}
(a_{i_1,j} - a_{i_1,\alpha P+j}) &= \left(\left\lfloor \frac{j-1}{p_{i_1}} \right\rfloor + \rho_{i_1} - \left\lfloor \frac{\alpha P+j-1}{p_{i_1}} \right\rfloor - \rho_{i_1} \right) \\
&= \left(\left\lfloor \frac{j-1}{p_{i_1}} \right\rfloor - \frac{\alpha P}{p_{i_1}} - \left\lfloor \frac{j-1}{p_{i_1}} \right\rfloor \right) \\
&= \left(\left\lfloor \frac{j-1}{p_{i_2}} \right\rfloor - \frac{\alpha P}{p_{i_2}} - \left\lfloor \frac{j-1}{p_{i_2}} \right\rfloor \right) \\
&= \left(\left\lfloor \frac{j-1}{p_{i_2}} \right\rfloor + \rho_{i_2} - \left\lfloor \frac{\alpha P+j-1}{p_{i_2}} \right\rfloor - \rho_{i_2} \right) \\
&= (a_{i_2,j} - a_{i_2,\alpha P+j}) \bmod q.
\end{aligned}$$

Similarly, if $h+1 \leq i_1 \leq i_2 \leq n, 1 \leq j \leq P$, and $0 \leq \alpha \leq q-1$, then

$$(a_{i_1,j} - a_{i_1,\alpha P+j}) \bmod q = (a_{i_2,j} - a_{i_2,\alpha P+j}) \bmod q.$$

This shows that if column j is h-bichromatic, then so is column $\alpha P + j$. Next, we show that if j is h-bichromatic then one and only one of the columns $j, P + j, \ldots, qP + j$ is a monochromatic column.

Let $j \in \{1, 2, \ldots, P\}$ and assume column j is h-bichromatic. Let us denote by the order pair (c_u, c_l) the entries of an h-bichromatic column where $a_{ij} = c_u$ for all $i \in \{1, \ldots, h\}$ and $a_{ij} = c_l$ for all $i \in \{h+1, \ldots, n\}$. Observe that if $c_u = c_l$ then we have a monochromatic column. Let $r_u = P/p_1 \bmod q$ and $r_l = P/p_n \bmod q$. Since $p_1 \equiv p_i \bmod q$, we have that $r_u \equiv P/p_i \bmod q$ for $i \in \{1, \ldots, h\}$. The value of r_u when $i \in \{0, \ldots, q-1\}$ then tells us how the color is changing as we move P columns over. In a similar way, r_l governs how the lower part changes.

Now, the solutions to the congruence $c_l + \alpha r_l \equiv c_u + \alpha r_u (\bmod q)$ will tell us which columns in the list $j, j+p, \ldots, j+(q-1)P$ are monochromatic. By Lemma 2.1, $r^{q-1} \equiv 1 (\bmod q)$, so

$$\begin{aligned} c_l - c_u + \alpha r_l - \alpha r_u &\equiv 0 \bmod q \\ c_l - c_u + \alpha(r_l - r_u) &\equiv 0 \bmod q \\ (c_l - c_u) r^{q-2} + \alpha &\equiv 0 \bmod q \\ \alpha &\equiv (c_u - c_l) r^{q-2} \bmod q. \end{aligned}$$

Hence, there is one solution which shows that one and only one of the columns $j, j+P, \ldots, j+(q-1)P$ is monochromatic. So, the number of h-bichromatic columns in the first P columns is the number of monochromatic columns in the whole matrix.

Now we count the number of h-bichromatic columns in the first P columns. Similarly to what was done in the proof of Proposition 3.1, it can be shown that for any column j, we have $k_{ij} \equiv j - b_1 a_{1,j} + b_1 \rho_i \bmod q$, for $i \in \{1, \ldots, h\}$ and

$k_{ij} \equiv j - b_2 a_{n,j} + b_2\rho_i \bmod q$, for $i \in \{h+1, \ldots, n\}$.

Let $1 \leq j \leq P, \beta_1 = j - b_1 a_{1j}$, and $\beta_2 = j - b_2 a_{n,j}$. Then, column j is h-bichromatic if and only if $k_{ij} \equiv \beta_1 + b_1\rho_i (\bmod q)$ for all $i \in \{1, \ldots, h\}$ and $k_{ij} \equiv \beta_2 + b_2\rho_i (\bmod q)$ for all $i \in \{h+1, \ldots, n\}$. So, setting $x_i = \frac{k_{ij} - (\beta_1 + b_1\rho_i) \bmod q}{q}$, we have that $1 \leq qx_i + (\beta_1 + b_1\rho_i) \bmod q \leq p_i$, or equivalently, $\frac{1-(\beta_1+b_1\rho_i) \bmod q}{q} \leq x_i \leq \frac{p_i - (\beta_1 + b_1\rho_i) \bmod q}{q} = \frac{p_i - 1}{q} - \frac{(\beta_1 + b_1\rho_i) \bmod q - 1}{q}$, where x_i is an integer. Now, by Lemma 2.1, the number of integer solutions to

$$\left\lceil \frac{1 - (\beta_1 + b_1\rho_i) \bmod q}{q} \right\rceil \leq x \leq \left\lfloor \frac{p_i - b_1}{q} - \frac{p_i - (\beta_1 + b_1\rho_i) \bmod q}{q} \right\rfloor$$

is

$$\begin{aligned} &\left\lfloor \frac{p_i - b_1}{q} - \frac{(\beta_1 + b_1\rho_i) \bmod q - b_1}{q} \right\rfloor - \left\lceil \frac{1 - (\beta_1 + b_1\rho_i) \bmod q}{q} \right\rceil + 1 \\ &= \frac{p_i - b_1}{q} + \left\lfloor \frac{b_1 - (\beta_1 + b_1\rho_i) \bmod q}{q} \right\rfloor + \left\lfloor \frac{(\beta_1 + b_1\rho_i) \bmod q - 1}{q} \right\rfloor + 1, \end{aligned}$$

and therefore the number of solutions for k_{ij} for $i \in \{1, 2, \ldots, h\}$. Similarly, the number of solutions to

$$\left\lceil \frac{1 - (\beta_2 + b_2\rho_i)}{q} \right\rceil \leq x \leq \left\lfloor \frac{p_i - b_2}{q} - \frac{(\beta_2 + b_2\rho_i) \bmod q - b_2}{q} \right\rfloor$$

is

$$\begin{aligned} &\left\lfloor \frac{p_i - b_2}{q} - \frac{(\beta_2 + \rho_i b_2) \bmod q - b_2}{q} \right\rfloor - \left\lceil \frac{1 - (\beta_2 + \rho_i b_2) \bmod q}{q} \right\rceil + 1 \\ &= \frac{p_i - b_2}{q} + \left\lfloor \frac{b_2 - (\beta_2 + \rho_i b_2) \bmod q}{q} \right\rfloor + \left\lfloor \frac{(\beta_2 + \rho_i b_2) \bmod q - 1}{q} \right\rfloor + 1 \end{aligned}$$

and therefore the number of solutions for k_{ij} for $i \in \{h+1, \ldots, n\}$. Similar to the proof in Proposition 3.1, by the CRT we know there is a one-to-one correspondence between $j \in \{1, \ldots, P\}$ and $(k_{1j}, \ldots, k_{nj})$. Summing over the q possibilities for both β_1 and β_2 and multiplying the number of solutions for each row, we obtain the count of h-bichromatic columns in the first P columns of A and hence the desired expression for the count of monochromatic columns in A:

$$\begin{aligned} N(A) &= \sum_{\beta_1=0}^{q-1}\sum_{\beta_2=0}^{q-1}\left(\prod_{i=1}^{t_1}\frac{p_i-b_1}{q}+\left\lfloor\frac{b_1-\beta_1 \bmod q}{q}\right\rfloor+\left\lfloor\frac{\beta_1 \bmod q+q-1}{q}\right\rfloor\right) \\ &\times\left(\prod_{i=t_1+1}^{t_2}\frac{p_i-b_1}{q}+\left\lfloor\frac{b_1-(\beta_1+1) \bmod q}{q}\right\rfloor+\left\lfloor\frac{(\beta_1+1) \bmod q+q-1}{q}\right\rfloor\right) \\ &\vdots \\ &\times\left(\prod_{i=t_{q-1}}^{t_q}\frac{p_i-b_1}{q}+\left\lfloor\frac{b_1-(\beta_1+q-1) \bmod q}{q}\right\rfloor+\left\lfloor\frac{(\beta_1+q-1) \bmod q+q-1}{q}\right\rfloor\right) \\ &\times\left(\prod_{i=t_q}^{t_{q+1}}\frac{p_i-b_2}{q}+\left\lfloor\frac{b_2-\beta_2 \bmod q}{q}\right\rfloor+\left\lfloor\frac{\beta_2 \bmod q+q-1}{q}\right\rfloor\right) \\ &\vdots \\ &\times\left(\prod_{i=t_{2q-1}}^{n}\frac{p_i-b_2}{q}+\left\lfloor\frac{b_2-(\beta_2+q-1) \bmod q}{q}\right\rfloor+\left\lfloor\frac{(\beta_2+2) \bmod q+q-1}{q}\right\rfloor\right) \end{aligned}$$

$$= \sum_{\beta_1=0}^{q-1}\sum_{\beta_2=0}^{q-1}\prod_{s=1}^{2}\prod_{\rho=0}^{q-1}\prod_{i=t_{q(s-1)+\rho}+1}^{i=t_{q(s-1)+\rho+1}}\left(\frac{p_i-b_s}{q}+\left\lfloor\frac{b_s-(\beta_s+\rho b_s) \bmod q}{q}\right\rfloor+\left\lfloor\frac{(\beta_s+\rho b_s) \bmod q+q-1}{q}\right\rfloor\right). \quad \square$$

For our next case, we assume that we know the number of monochromatic columns for a color matrix B. Now, we will add another p_i that is congruent to q. So, our possible colors are $0, 1, \ldots, q-1$. Since we assume that the p_i are pairwise coprime, there is at most one. The following proposition will compute the number of monochromatic columns.

Proposition 3.3. *Let n be a positive integer such that $n \geq 2$, q be prime, and $p_1, \ldots, p_n$ be positive, pairwise coprime integers such that $q \mid p_n$. Let B be a color matrix associated with pairwise coprime integers $p_1, ..., p_{n-1}$. Let A be a color matrix associated with pairwise coprime integers $p_1, \ldots, p_n$ where the first $n-1$ rows of A are p_n copies of matrix B. Then*

$$N(A) = \frac{p_n}{q}N(B)$$

Proof. Let $P = p_1p_2\cdots p_n$. Denote the initial color of row i in the A matrix by ρ_i. The color entries of the A matrix is given by $a_{ij} = \left(\left\lfloor \frac{j-1}{p_i} \right\rfloor + \rho_i\right) \bmod q$, and, by definition, $b_{ij} = \left\lfloor \frac{j-1}{p_i} \right\rfloor$ and $k_{ij} = j - b_{ij}p_i$. We will show that if column j is $(n-1)$-bichromatic, then a column some multiple of P away from j is also $(n-1)$-bichromatic.

Let $1 \le i \le n-1, 1 \le j \le P$, and $0 \le \alpha \le q-1$. Since $q|\frac{P}{p_i}$,

$$\begin{aligned}
(a_{i,j} - a_{i,\alpha P+j}) \bmod q &= \left(\left\lfloor \frac{j-1}{p_i} \right\rfloor + \rho_i - \left\lfloor \frac{\alpha P + j - 1}{p_i} \right\rfloor - \rho_i\right) \bmod q \\
&= \left(\left\lfloor \frac{j-1}{p_i} \right\rfloor - \frac{\alpha P}{p_i} - \left\lfloor \frac{j-1}{p_i} \right\rfloor\right) \bmod q \\
&= \left(-\frac{\alpha P}{p_i}\right) \bmod q \\
&= 0 \bmod q.
\end{aligned}$$

Let $r = (-P/p_n) \bmod q$. Note that $r \neq 0 \pmod q$. So,

$$\begin{aligned}
(a_{n,j} - a_{n,\alpha P+j}) \bmod q &= \left(\left\lfloor \frac{j-1}{p_n} \right\rfloor + \rho_n - \left\lfloor \frac{\alpha P + j - 1}{p_n} \right\rfloor - \rho_n\right) \bmod q. \\
&= \left(\left\lfloor \frac{j-1}{p_n} \right\rfloor - \frac{\alpha P}{p_n} - \left\lfloor \frac{j-1}{p_n} \right\rfloor\right) \bmod q \\
&= \left(-\frac{\alpha P}{p_n}\right) \bmod q \\
&= (\alpha r) \bmod q.
\end{aligned}$$

From what we have just shown, $\alpha P + j$ is also $(n-1)$-bichromatic.

Let $j \in \{1, 2, \ldots, P\}$, and assume column j is $(n-1)$-bichromatic. In a similar fashion to the proof of Proposition 3.2, one and only one of the $(n-1)$-bichromatic columns $j, P+j, ..., \alpha P + j$ is monochromatic. So, the number of $(n-1)$-bichromatic columns in the first P columns is the number of monochromatic columns in the whole matrix. Then the first $n-1$ rows of A can be viewed as p_n copies of matrix B side by side. So the number of $(n-1)$-bichromatic columns in A is $p_nN(B)$ and hence $N(A) = (p_n/q)N(B)$. □

We have shown three specific cases where the number of monochromatic columns can be determined. The first proposition counted the number of monochromatic columns when all $p_1, \ldots, p_n$ were congruent to $s \bmod q$. The second proposition counted the number of monochromatic columns where all $p_1, \ldots, p_n$ were congruent to either $b_1 \pmod q$ or $b_2 \pmod q$. In the third proposition, we assumed that we knew the number of monochromatic columns in some matrix B, and then we found the number of monochromatic columns in matrix A, which was a matrix generated by adding p_n to matrix B. In the propositions, we found that when we have a bichromatic column, then as we keep moving P columns over, eventually our colors will line up and we get a monochromatic column. So, we were able to count the number of bichromatic columns in the first P columns and that was equal to the number of monochromatic columns in the whole matrix. This technique does not work for all cases. If we have $p_1, \ldots, p_n$ that are congruent modulo q to b_1, b_2, b_3, then we would have to come up with another method. The following is a counterexample that shows that if we have p_1, p_2, p_3 that are not congruent modulo q, then as we move P columns over throughout the matrix we will never get a monochromatic column. Hence, we are unable to count the number of columns with three colors in the first P columns and say that there will be that number of monochromatic columns in the entire matrix.

Example 3.1. Let $(5; 2, 3, 11; 0, 1, 3; 1, 2)$ be a color matrix. Then $P = 66$ and we have a 3×330 matrix. A column is monochromatic if $a_{1j} = a_{2j} = a_{3j}$.
The first few columns of our color matrix would look like

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 0 & 0 \ldots \\ 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 \ldots \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 0 \ldots \end{pmatrix}.$$

As used in the propositions, $r_i = P/p_i \bmod q$. For our three rows,

$r_1 = 66/2 \bmod 5 = 3 \bmod 5$

$r_2 = 66/3 \bmod 5 = 2 \bmod 5$

$r_3 = 66/11 \bmod 5 = 1 \bmod 5.$

The values for r_1, r_2, and r_3 tell us how the color is changing as we move P columns over. For example, in row one each color is repeated twice. Over the first P columns, our color would change 33 times and 33 modulo 5 is 3, which means that our color changes three times as we move P columns over. The matrix below shows the columns 1, 67, 133, 199, and 265, respectively, which are each a multiple of P away from our first column. We see that none of these columns are monochromatic.

$$\begin{pmatrix} 0\ldots & 3\ldots & 1\ldots & 4\ldots & 2\ldots \\ 1\ldots & 3\ldots & 0\ldots & 2\ldots & 4\ldots \\ 3\ldots & 4\ldots & 0\ldots & 1\ldots & 2\ldots \end{pmatrix}$$

In our propositions, we showed that when we have a bichromatic column in one of the first P columns, then some column $j + \alpha P$ will be monochromatic. So, the number of bichromatic columns in the first P columns would be equal to the number of monochromatic columns in the matrix. We see in this example that there is not a column $j + \alpha P$ that is monochromatic. Therefore, we cannot find the number of columns composed of three colors in the first P columns and say that there will be that number of monochromatic columns in the entire matrix.

Chapter 4

Conclusion

In conclusion, the Monochromatic Column Problem was partially solved for a prime number of colors. First, a solution was provided for the case where all p_i are congruent to $s(\bmod q)$. Then, a solution was provided where all p_i are congruent to $b_1(\bmod q)$ or $b_2(\bmod q)$. Finally, a solution was provided where we assume we know the number of monochromatic columns in some matrix B and we add p_n that is congruent to q. However, we cannot generalize this for all scenarios because a counterexample was given demonstrating that the method used in the three propositions fails. Finding a general solution for some prime number of colors is an open problem. For a composite number of colors, the problem is completely open.

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