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Positive Symmetric Solutions Of A Boundary Value Problem With Dirichlet Boundary Conditions

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POSITIVE SYMMETRIC SOLUTIONS OF A BOUNDARY VALUE PROBLEM WITH DIRICHLET BOUNDARY CONDITIONS

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BY

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DEDICATION

This thesis is dedicated to my parents.

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ABSTRACT

We apply a recent extension of a compression-expansion fixed point theorem of function type to a second order boundary value problem with Dirichlet boundary conditions. We show the existence of positive symmetric solutions of this boundary value problem.

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1 INTRODUCTION

There have been many studies in the past on finding positive solutions of boundary value problems. In 1979, Leggett and Williams [14] came up with fixed point theorems which guaranteed the existence of solutions of the boundary value problems. The first theorem helps find a fixed point, the second one guarantees the existence of at least two fixed points, and the other two theorems guarantee the existence of at least three fixed points. The following theorem is the original Leggett-Williams theorem which guarantees the existence of a fixed point.

Theorem 1.1 (Leggett-Williams [14])**.** *Consider a Banach space B, the cone K, and the operator T. For α a positive concave functional, we define the following subsets of K:*

$$K_c = \{x \in K : \|x\| < c\},$$

$$K(\alpha, b, d) = \{x \in K : b \leq \alpha(x), \|x\| \leq d\}, \text{ and}$$

Suppose that $A : K_c \to K$ is completely continuous and suppose there exist a concave positive functional α with $\alpha(x) \leq \|x\|$ $(x \in K)$ and numbers $b > a > 0 (b \leq c)$ satisfying the following conditions:

(1) $\{x \in S(\alpha, a, b) : \alpha(x) > a\} \neq \emptyset$, and $\alpha(Ax) > a$ if $x \in S(\alpha, a, b)$;

(2) $Ax \in K_c$ if $x \in S(\alpha, a, c)$; and

(3) $\alpha(Ax) > a$ for all $x \in S(\alpha, a, c)$ with $\|Ax\| > b$.

Then A has a fixed point x in $S(\alpha, a, c)$.

Avery and other co-authors have generalized the Leggett-Williams theorem over the years, modifying the conditions, and have proven their theorem guarantees the existence of a positive solution. Some recent works of Avery, Anderson, and Henderson [10, 13] have also proven that there exists at least one positive solution for a right focal boundary value problem. The compression-expansion fixed point theorem [10] helps find the fixed point when there is nonnegative continuous concave functionals and nonnegative continuous convex functions on $\mathcal{P}$, and when operator T is completely continuous. Some of the applications of the extensions of the Leggett-Williams fixed point theorem and the compression-expansion fixed point theorem are stated below.

In 2012, Altwaty and Eloe [2] have also used an extension of the original Leggett-Williams fixed point theorem to show the existence of at least one positive solution of a two point boundary value problem of the kth order differential equation.

In 2012, Avery, Eloe, and Henderson [8], used an extension of Leggett-Williams type fixed point theorem to study a two point boundary value problem for an ordinary differential equation that has a fourth order

$$x^{(iv)}(t) = f(x(t)), \quad t \in [0, 1],$$

$$x^{(i)}(0) = 0,\ x^{(i)}(1) = 0, \quad i = 0, 1.$$

Similarly, in 2013, Altwaty and Eloe [1] used an extension of a Leggett-Williams in order to show that operator has at least one positive solution. They worked on two point conjugate boundary value problem

$$(-1)^n x^{(2n)}(t) = f(x(t)), \quad t \in [0, 1],$$

$$x^{(i)}(0) = 0,\ x^{(i)}(1) = 0, \quad i = 0, 1, ...n$$

where n is a positive integer.

In 2016, Avery, Anderson, and Henderson [10] provided a modified version of the fixed point theorem which is known as the extension of the compression-expansion fixed point theorem. They used a compression-expansion fixed point theorem to show the existence of a positive solution for a right focal boundary value problem

$$x''(t) + f(x(t)) = 0, \quad t \in (0, 1),$$

$$x(0) = 0 = x'(1),$$

where $f : \mathbb{R} \to [0, \infty)$ is continuous. If x is a fixed point of the operator T defined by

$$Tx(t) := \int_0^1 G(t, s) f(x(s)) ds,$$

where $G(t, s) = \min\{t, s\}$, $(t, s) \in [0, 1] \times [0, 1]$, then x is a solution of the boundary value problem mentioned above.

In this paper, we will show the existence of a positive symmetric solution of the boundary value problem by using a compression-expansion fixed point theorem of function type [10]. Here, we have a second order boundary value problem with Dirichlet boundary value conditions. We will apply the fixed point theorem in order to show the existence of a positive symmetric solution to the differential equation

$$x''(t) + f(x(t)) = 0, \quad t \in (0, 1), \tag{1.1}$$

when it has the Dirichlet boundary conditions

$$x(0) = 0 = x(1), \tag{1.2}$$

where $f : \mathbb{R} \to [0, \infty)$ is continuous.

In Chapter 2, we have definitions and fixed point theorem. Under the fixed point theorem, we have extension of compression-expansion fixed point theorem. In Chapter 3, we have application of the extension of compression-expansion fixed point theorem and preliminaries. In addition to that, we have the properties of the Green's function that we use throughout the paper, and the theorem we came up with. There are different conditions presented in the theorem, and if all these conditions are satisfied then we will show that our operator T has a fixed point $x^* \in A(\beta, b, \alpha, a)$. In other words, we will show the existence of a positive symmetric solution of (1.1), (1.2). We are motivated to work on this problem because of some of the recent works of Avery, Anderson, and Henderson [10, 13]. For more examples of Avery type fixed point theorems, see [3, 4, 5, 6, 7, 9, 11, 12].

2 THE FIXED POINT THEOREM

2.1 Definitions

Let us begin with some basic definitions.

Definition 1. A complete normed vector space is called a *Banach space*.

Definition 2. Let E be a real Banach space. A nonempty closed convex set $\mathcal{P} \subset E$ is called a *cone* if for all $x \in \mathcal{P}$ and $\lambda \geq 0$, $\lambda x \in \mathcal{P}$, and if $x,\ -x \in \mathcal{P}$ then $x = 0$.

Definition 3. A map α is said to be a *nonnegative continuous concave functional* on a cone $\mathcal{P}$ of a real Banach space E if $\alpha : \mathcal{P} \to [0, \infty)$ is continuous and if

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in \mathcal{P}$ and $t \in [0,1]$. Similarly, we say the map β is a *nonnegative continuous convex functional* on a cone $\mathcal{P}$ of a real Banach space E if $\beta : \mathcal{P} \to [0, \infty)$ is continuous and if

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in \mathcal{P}$ and $t \in [0,1]$.

Definition 4. An operator is called *completely continuous* if it is continuous and maps bounded sets into sets whose closures are compact.

2.2 The Fixed Point Theorem

Definition 5. Let A be a relatively open subset of a cone $\mathcal{P}$, b and c positive real numbers, α a concave functional on $\mathcal{P}$, and β a convex functional on $\mathcal{P}$. We say that the open set

$$\mathcal{P}(\beta, b) = \{x \in \mathcal{P} : \beta(x) < b\}$$

is a *functional wedge* on a cone, that a set of the form

$$\mathcal{P}(\beta, b, \alpha, c) = \{x \in \mathcal{P} : c < \alpha(x) \text{ and } \beta(x) < b\}$$

is a *functional frustum* of a cone, and if A is an open subset of the cone $\mathcal{P}$, we say that

$$A(\beta, b, \alpha, c) = \{x \in A : c < \alpha(x) \text{ and } \beta(x) < b\}$$

is an interval of functional type.

Theorem 2.1 (Extension of Compression-Expansion Fixed Point Theorem [10]). *Suppose $\mathcal{P}$ is a cone in a real Banach space E, A is a relatively open subset of $\mathcal{P}$, α and ψ are nonnegative continuous concave functionals on $\mathcal{P}$, β and θ are nonnegative continuous convex functionals on $\mathcal{P}$, and $T : \mathcal{P} \to \mathcal{P}$ is a completely continuous operator. If there exist nonnegative numbers a, b, c, and d such that*

(A1) $A(\beta, b, \alpha, a)$ is bounded, $A(\beta, b, \alpha, a) \cap A(\theta, c, \psi, d) \neq \emptyset$, and if $x \in \partial A \cap \overline{\mathcal{P}(\beta, b, \alpha, a)} = \emptyset$ then $Tx \neq x$;

(A2) if $x \in \partial A(\beta, b, \alpha, a)$ with $\alpha(x) = a$ and either $\theta(x) \leq c$ or $\theta(Tx) > c$, then $\alpha(Tx) > a$;

(A3) if $x \in \partial A(\beta, b, \alpha, a)$ with $\beta(x) = b$ and either $\psi(Tx) < d$ or $\psi(x) \geq d$, then $\beta(Tx) < b$; then T has a fixed point $x^* \in A(\beta, b, \alpha, a)$.

3 POSITIVE SYMMETRIC SOLUTIONS

3.1 Preliminaries

Define the Banach space

$$E = C^{(1)}[0,1] = \{x : [0,1] \to \mathbb{R} : x' \text{ is continuous}\}$$

with norm

$$||x|| = \max_{t\in[0,1]} |x(t)| + \max_{t\in[0,1]} |x'(t)|.$$

Define the cone $\mathcal{P} \subseteq E$ by

$$\mathcal{P} = \{x \in E : x(1-t) = x(t),\ x \text{ is nondecreasing on } [0,1/2],$$

$$\text{nonnegative on } [0,1], \text{and concave on } [0,1]\}.$$

The Green's function for $-x'' = 0$, (1.2), is given by

$$G(t,s) = \begin{cases} t(1-s), & 0 < t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

Therefore, x is a solution of (1.1), (1.2) if and only if x solves the integral equation

$$x(t) = \int_0^1 G(t,s) f(x(s))\, ds, \quad t \in [0,1].$$

Notice the derivative of the Green's function with respect to t is given by

$$\frac{\partial}{\partial t} G(t,s) = \begin{cases} (1-s), & 0 < t \le s \le 1, \\ -s, & 0 \le s \le t \le 1. \end{cases}$$

Lemma 3.1. *The function G satisfies the following properties.*

1. $G(t,s) \in C([0,1] \times [0,1])$ *with* $G(t,s) \geq 0$ *for all* $(t,s) \in [0,1] \times [0,1]$.

2. $G(1-t, 1-s) = G(t,s)$ *for all* $(t,s) \in [0,1] \times [0,1]$.

3. *For any* $y, w \in [0, 1/2]$ *with* $y \leq w$,

$$yG(w,s) \leq wG(y,s).$$

Define the operator $T : E \to E$ by

$$Tx(t) = \int_0^1 G(t,s) f(x(s)) ds, \quad t \in [0,1].$$

If x is a fixed point of T, i.e. if $Tx = x$, then x solves (1.1), (1.2). In [2], it is shown that $T : \mathcal{P} \to \mathcal{P}$ and that T is completely continuous.

Define the nonnegative continuous functionals by

$$\beta(x) = \max_{t \in [0,1/2]} x(t) = x(1/2),$$

$$\psi(x) = x'(1/8),$$

and

$$\alpha(x) = \min_{t \in [1/8,1/2]} x(t) = x(1/8).$$

3.2 Positive Symmetric Solutions

Theorem 3.2. *Choose* k_1, k_2, k_3, k_4, *and* k_5 *positive so that*

(a) $\frac{128}{7} k_4 < 8k_3$,

(b) $k_1 < \left(\frac{83}{40}\right) \left(k_3 - \frac{9}{256} k_2\right)$,

(c) $16 \left(k_4 - \left(\frac{192}{913}\right) k_1\right) < k_3 - k_5$,

(d) $\frac{4}{55}\left(k_3 - \left(\frac{9}{256}\right)k_2\right) > k_4 - \left(\frac{192}{913}\right)k_1$,

(e) $k_2 > 64\left(k_4 - \left(\frac{192}{913}\right)k_1\right)$,

(f) $k_2 > \left(\frac{2048}{913}\right)k_1$,

(g) $k_3 - k_5 > \left(\frac{512}{913}\right)k_1$,

(h) $k_3 > \frac{1}{8}\left(k_5 + 8k_1\right)$, *and*

(i) $k_4 < (\frac{1}{8})(k_5 + 8k_1)$.

Let $f : [0,\infty) \to [0,\infty)$ *be a continuous function and* b *be a positive real number such that*

(H1) $f(w) > \frac{4{,}096}{913}k_1 b$ *for* $w \in [k_4 b, k_3 b]$,

(H2) $f(w) < 2k_2 b$ *for* $w \in [0, k_3 b]$,

(H3) $f(w) < 8(k_3 b - k_5 b)$ *for* $w \in [0, k_3 b]$,

(H4) $f(w) > 128\left(k_4 b - \frac{192}{913}k_1 b\right)$ *for* $w \in [0, k_4 b]$, *and*

(H5) $f(w) < \frac{512}{55}\left(k_3 b - \frac{9}{256}k_2 b\right)$ *for* $w \in [\frac{1}{8}b(k_5 + 8k_1), k_3 b]$.

Then T *has a fixed point* $x^* \in A(\beta, k_3 b, \alpha, k_4 b)$; *i.e.* (1.1), (1.2) *has a solution* $x^* \in A(\beta, k_3 b, \alpha, k_4 b)$.

Proof. Let

$$A = \{x \in \mathcal{P} : x(3/16) - x(1/8) > k_1 b \text{ and } x'(0) < k_2 b\}.$$

Consider $A(\beta, k_3 b, \alpha, k_4 b) = \{x \in A : k_4 b < x(t) < k_3 b \text{ for } t \in [1/8, 1/2]\}$, and

$$A(\beta, k_3 b, \psi, k_5 b) = \{x \in A : x(t) < k_3 b \text{ for } t \in [0, 1/2] \text{ and } x'(1/4) > k_5 b\}.$$

(A1) We show $A(\beta, k_3 b, \alpha, k_4 b)$ is bounded. Let $x \in A(\beta, k_3 b, \alpha, k_4 b)$. Then

$$\begin{aligned}||x|| &= \max_{t\in[0,1]} |x(t)| + \max_{t\in[0,1]} |x'(t)| \\ &= x(1/2) + x'(0) \leq k_3 b + k_2 b \\ &= (k_3 + k_2) b.\end{aligned}$$

Therefore, $A(\beta, k_3 b, \alpha, k_4 b)$ is bounded open subset of $\mathcal{P}$.

Next, we show $A(\beta, k_3 b, \alpha, k_4 b) \cap A(\beta, k_3 b, \psi, k_5 b) \neq \emptyset$. Choose K such that $K \leq 8k_3$, $K \geq \frac{128}{7} k_4$, and $K \geq \frac{8}{3} k_5$. Let

$$w_0(t) = \int_0^1 G(t, s) Kb \, ds.$$

Then $x \in \mathcal{P}$,

$$\begin{aligned}\beta(w_0) = w_0\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2}, s\right) Kb \, ds \\ &= Kb \int_0^{1/2} s \, ds \\ &= \frac{Kb}{8} \\ &\leq k_3 b.\end{aligned}$$

In addition,

$$\begin{aligned}\alpha(w_0) = w_0\left(\frac{1}{8}\right) &= \int_0^1 G\left(\frac{1}{8}, s\right) Kb \, ds \\ &= \int_0^{1/8} s\left(1 - \frac{1}{8}\right) Kb \, ds + \int_{1/8}^1 \frac{1}{8}(1 - s) Kb \, ds \\ &= Kb\left(\frac{7}{1024} + \frac{49}{1024}\right) \\ &= \frac{7Kb}{128} \\ &\geq k_4 b.\end{aligned}$$

Lastly,

$$\begin{aligned}\psi(w_0) = w_0'\left(\frac{1}{8}\right) &= \int_0^{1/8} (-s)Kb\ ds + \int_{1/8}^1 (1-s)Kb\ ds \\ &= Kb\left(-\frac{1}{128} + \frac{49}{128}\right) \\ &= \frac{3Kb}{8} \\ &\geq k_5 b.\end{aligned}$$

Therefore,

$$w_0 \in A(\beta, k_3b, \alpha, k_4b) \cap A(\beta, k_3b, \psi, k_5b)$$

and

$$A(\beta, k_3b, \alpha, k_4b) \cap A(\beta, k_3b, \psi, k_5b) \neq \emptyset.$$

Finally, we show if $x \in \partial A \cap \overline{\mathcal{P}(\beta, k_3b, \alpha, k_4b)}$, then $Tx \neq x$. Let $x \in A(\beta, k_3b, \alpha, k_4b)$ with $Tx = x$. Then

$$x\left(\frac{1}{8}\right) > k_4b \text{ and } x\left(\frac{1}{2}\right) < k_3b.$$

Now,

$$\begin{aligned}x\left(\frac{3}{16}\right) - x\left(\frac{1}{8}\right) &= (Tx)\left(\frac{3}{16}\right) - (Tx)\left(\frac{1}{8}\right) \\ &= \int_0^{3/16} s\left(1-\frac{3}{16}\right) f(x(s))ds + \int_{3/16}^1 \frac{3}{16}(1-s)f(x(s))ds \\ &\quad - \int_0^{1/8} s\left(1-\frac{1}{8}\right) f(x(s))ds - \int_{1/8}^1 \frac{1}{8}(1-s)f(x(s))ds \\ &= \int_0^{1/8} -\frac{1}{16}sf(x(s))ds + \int_{1/8}^{3/16} \frac{1}{16}(15s-2)f(x(s))ds \\ &\quad + \int_{3/16}^1 \frac{11}{16}(1-s)f(x(s))ds \\ &= \int_0^{1/8} -\frac{1}{16}sf(x(s))ds + \int_{1/8}^{3/16} \frac{1}{16}(15s-2)f(x(s))ds \\ &\quad + \int_0^{13/16} \frac{11}{16}sf(x(s))ds\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{1/8} \frac{5}{8} s f(x(s)) ds + \int_{1/8}^{3/16} \frac{1}{8}(13s-1) f(x(s)) ds \\
&\quad + \int_{3/16}^{13/16} \frac{11}{16} s f(x(s)) ds \\
&> \int_{1/8}^{3/16} \frac{1}{8}(13s-1) f(x(s)) ds + \int_{3/16}^{13/16} \frac{11}{16} s f(x(s)) ds.
\end{aligned}
$$

If $s \in [1/8, 13/16]$, $x(s) \in [k_4 b, k_3 b]$. By (H1), $f(w) > \frac{4{,}096}{913} k_1 b$ for $w \in [k_4 b, k_3 b]$. Thus, $f(x(s)) > \frac{4{,}096}{913} k_1 b$ for $s \in [1/8, 3/16]$. So,

$$
\begin{aligned}
x\left(\frac{3}{16}\right) - x\left(\frac{1}{8}\right) &> \int_{1/8}^{3/16} \frac{1}{8}(13s-1)\frac{4,096}{913} k_1 b \ ds + \int_{3/16}^{13/16} \frac{11}{16} s \frac{4,096}{913} k_1 b \ ds \\
&= \frac{4,096}{(8)(913)} k_1 b \int_{1/8}^{3/16} (13s-1) \ ds + \frac{(11)(4,096)}{(16)(913)} k_1 b \int_{3/16}^{13/16} s \ ds \\
&= \frac{4,096}{7,304} k_1 b \left(\frac{13s^2}{2} - s\right) \Big|_{1/8}^{3/16} + \frac{45,056}{14,608} k_1 b \left(\frac{s^2}{2}\right) \Big|_{3/16}^{13/16} \\
&= \frac{512}{913} k_1 b \left(\frac{13s^2}{2} - s\right) \Big|_{1/8}^{3/16} + \frac{256}{83} k_1 b \left(\frac{s^2}{2}\right) \Big|_{3/16}^{13/16} \\
&= \frac{33}{913} k_1 b + \frac{640}{664} k_1 b \\
&= \frac{364 k_1 b + 7,040 k_1 b}{7,304} \\
&= \frac{7,304}{7,304} k_1 b \\
&= k_1 b.
\end{aligned}
$$

Also, $x'(0) = (Tx)'(0) = \int_0^1 (1-s) f(x(s)) \ ds$. If $s \in [0,1]$, $x(s) \in [0, k_3]$. By (H2), $f(w) < 2k_2 b$ for $w \in [0, k_3 b]$. Thus, $f(x(s)) < 2k_2 b$ for $s \in [0,1]$. So,

$$
\begin{aligned}
x'(0) &< \int_0^1 (1-s) 2k_2 b \ ds \\
&= 2k_2 b \int_0^1 (1-s) \ ds \\
&= 2k_2 b \left(s - \frac{s^2}{2}\right) \Big|_0^1
\end{aligned}
$$

$$
\begin{aligned}
&= 2k_2 b \left(1 - \frac{1}{2}\right) \\
&= 2k_2 b \left(\frac{1}{2}\right) \\
&= k_2 b.
\end{aligned}
$$

Thus, $x \notin \partial A(\beta, k_3 b, \alpha, k_4 b)$. So by contrapositive, if $x \in \partial A(\beta, k_3 b, \alpha, k_4 b)$, then $Tx \neq x$.

(A2) Let $x \in \partial A(\beta, k_3 b, \alpha, k_4 b)$ with $\alpha(x) = k_4 b$. Now $x\left(\frac{1}{8}\right) = k_4 b$.

$$
\begin{aligned}
\alpha(Tx) &= (Tx)\left(\frac{1}{8}\right) \\
&= \int_0^1 G\left(\frac{1}{8}, s\right) f(x(s))ds \\
&= \int_0^{1/8} s\left(1 - \frac{1}{8}\right) f(x(s))ds + \int_{1/8}^1 \frac{1}{8}(1-s) f(x(s))ds \\
&= \int_0^{1/8} \frac{7}{8} s f(x(s))ds + \int_{1/8}^1 \frac{1}{8}(1-s) f(x(s))ds \\
&= \int_0^{1/8} \frac{7}{8} s f(x(s))ds + \int_0^{7/8} \frac{1}{8} s f(x(s))ds \\
&= \int_0^{1/8} s f(x(s))ds + \int_{1/8}^{7/8} \frac{1}{8} s f(x(s))ds.
\end{aligned}
$$

If $s \in [0, 1/8]$, $x(s) \in [0, k_4 b]$. By (H4), $f(w) > 128\left(k_4 b - \frac{192}{913} k_1 b\right)$ for $w \in [0, k_4 b]$. Thus, $f(x(s)) > 128\left(k_4 b - \frac{192}{913} k_1 b\right)$ for $s \in [0, k_4 b]$. Also, if $s \in [1/8, 7/8]$, $x(s) \in [k_4 b, k_3 b]$. By (H1), $f(w) > \frac{4{,}096}{913} k_1 b$ for $w \in [k_4 b, k_3 b]$. Thus, $f(x(s)) > \frac{4{,}096}{913} k_1 b$ for $s \in [1/8, 7/8]$. Therefore,

$$
\begin{aligned}
\alpha(Tx) &> \int_0^{1/8} s\left(128\left(k_4 b - \frac{192}{913} k_1 b\right)\right)\, ds + \int_{1/8}^{7/8} s\left(\frac{4,096}{913} k_1 b\right)\, ds \\
&= 128\left(k_4 b - \frac{192}{913} k_1 b\right) \int_0^{1/8} s\, ds + \frac{4,096}{913} k_1 b \int_{1/8}^{7/8} \frac{1}{8} s\, ds \\
&= 128\left(k_4 b - \frac{192}{913} k_1 b\right) \left(\frac{s^2}{2}\right)\Big|_0^{1/8} + \frac{4,096}{913} k_1 b \left(\frac{s^2}{16}\right)\Big|_{1/8}^{7/8}
\end{aligned}
$$

$$
\begin{aligned}
&= 128\left(k_4 b - \frac{192}{913}k_1 b\right)\left(\frac{1}{128}\right) + \frac{4,096}{913}k_1 b\left(\frac{1}{16}\left(\frac{49}{64} - \frac{1}{64}\right)\right) \\
&= k_4 b - \frac{192}{913}k_1 b + \frac{4,096}{913}k_1 b\left(\frac{3}{64}\right) \\
&= k_4 b - \frac{192}{913}k_1 b + \frac{192}{913}k_1 b \\
&= k_4 b.
\end{aligned}
$$

(A3) Let $x \in \partial A(\beta, k_3 b, \alpha, k_4 b)$ with $\beta(x) = k_3 b$.

Suppose $\psi(x) \geq k_5 b$. Then $(x)'\left(\frac{1}{8}\right) \geq k_5 b$. Now $\psi(x) \geq k_5 b$ and x is increasing, so $x\left(\frac{1}{8}\right) \geq \frac{1}{8}k_5 b$, since

$$
\begin{aligned}
\frac{1}{8}k_5 b &\leq \frac{\psi(x)}{8} \\
&= \int_0^{1/8} x'\left(\frac{1}{8}\right) ds \\
&\leq \int_0^{1/8} x'(s) ds \\
&= x\left(\frac{1}{8}\right).
\end{aligned}
$$

Also, since $x \in A$,

$$
\begin{aligned}
x\left(\frac{3}{16}\right) &\geq x\left(\frac{1}{8}\right) + k_1 b \\
&\geq \frac{1}{8}b\left(k_5 + 8k_1\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\beta(Tx) &= \int_0^1 G\left(\frac{1}{2}, s\right) f(x(s))\ ds \\
&= \int_0^{1/2} s f(x(s))\ ds \\
&= \int_0^{3/16} s f(x(s))\ ds + \int_{3/16}^{1/2} s f(x(s))\ ds.
\end{aligned}
$$

If $s \in [0, 3/16]$, $x(s) \in [0, k_3 b]$. By (H2), $f(w) < 2k_2 b$ for $w \in [0, k_3 b]$. Thus, $f(x(s)) < 2k_2 b$ for $s \in [0, 3/16]$. Also, for $s \in [3/16, 1/2]$, $x(s) \in [\frac{1}{8}b(k_5 + 8k_1), k_3 b]$. By (H5), $f(w) < \frac{512}{55}\left(k_3 b - \frac{9}{256}k_2 b\right)$ for $w \in [\frac{1}{8}b(k_5 + 8k_1), k_3 b]$. Thus, $f(x(s)) <$

$\frac{512}{55}\left(k_3b - \frac{9}{256}k_2b\right)$ for $s \in [3/16, 1/2]$. Therefore,

$$\begin{aligned}
\beta(Tx) &< \int_0^{3/16} s\ 2k_2b\ ds + \int_{3/16}^{1/2} s\left(\frac{512}{55}\left(k_3b - \frac{9}{256}k_2b\right)\right)\ ds \\
&= 2k_2b\int_0^{3/16} s\ ds + \frac{512}{55}\left(k_3b - \frac{9}{256}k_2b\right)\int_{3/16}^{1/2} s\ ds \\
&= 2k_2b\left(\frac{9}{512}\right) + \frac{512}{55}\left(k_3b - \frac{9}{256}k_2b\right)\left(\frac{55}{512}\right) \\
&= \frac{9}{256}k_2b + k_3b - \frac{9}{256}k_2b \\
&= k_3b.
\end{aligned}$$

Next, suppose $\psi(Tx) < k_5b$. Now $(Tx)'\left(\frac{1}{8}\right) < k_5b$. So,

$$\begin{aligned}
\psi(Tx) &= (Tx)'\left(\frac{1}{8}\right) \\
&= \int_0^{1/8}(-s)f(x(s))ds + \int_{1/8}^{1}(1-s)f(x(s))ds \\
&= \int_0^{1/8}(-s)f(x(s))ds + \int_0^{7/8} sf(x(s))ds \\
&= \int_{1/8}^{7/8} sf(x(s))ds \\
&< k_5b.
\end{aligned}$$

Thus,
$$\begin{aligned}
\beta(Tx) &= (Tx)\left(\frac{1}{2}\right) \\
&= \int_0^1 sf(x(s))ds \\
&= \int_0^{1/8} sf(x(s))ds + \int_{1/8}^{7/8} sf(x(s))ds + \int_{7/8}^{1} sf(x(s))ds.
\end{aligned}$$

We also know that

$$\int_{7/8}^{1} sf(x(s))ds = \int_0^{1/8}(1-s)f(x(s))ds.$$

So,
$$\int_0^{1/8} sf(x(s))ds + \int_0^{1/8}(1-s)f(x(s))ds = \int_0^{1/8} f(x(s))ds.$$

Therefore, by combining them together, we get
$$\begin{aligned}\beta(Tx) &= \int_{1/8}^{7/8} sf(x(s))ds + \int_0^{1/8} f(x(s))\ ds \\ &< k_5b + \int_0^{1/8} f(x(s))\ ds\end{aligned}$$

If $s \in [0, 1/8]$, $x(s) \in [0, k_3b]$. By (H3), $f(w) < 8(k_3b - k_5b)$ for $w \in [0, k_3b]$. Thus, $f(x(s)) < 8\,(k_3b - k_5b)$ for $s \in [0, k_3b]$. So,

$$\begin{aligned}\beta(Tx) &< k_5b + \int_0^{1/8} 8\,(k_3b - k_5b)\ ds \\ &= k_5b + 8\,(k_3b - k_5b)\,(s)\Big|_0^{1/8} \\ &= k_5b + 8\,(k_3b - k_5b)\left(\frac{1}{8}\right) \\ &= k_5b + k_3b - k_5b \\ &= k_3b.\end{aligned}$$

Therefore, T has a fixed point $x^* \in A(\beta, k_3b, \alpha, k_4b)$, and (1.1), (1.2) has a positive symmetric solution $x^* \in A(\beta, k_3b, \alpha, k_4b)$. □

3.3 Example

Example 3.3. Example: Let $k_1 = 8380$, $k_2 = 230400$, $k_3 = 12140$, $k_4 = 1800$, and $k_5 = 80$. Then let us verify if the mentioned values satisfy the inequalities from (a) to (i).

(a) $\frac{128}{7}k_4 < 8k_3$. Then we have

$$\frac{128}{7}(1800) < 8(12140)$$
$$32914.2857 < 97120.$$

(b) $k_1 < (\frac{83}{40})\left(k_3 - \frac{9}{256}k_2\right)$. Similarly,

$$8380 < \frac{83}{40}\left(12140 - \frac{9}{256}(230400)\right) = 8383.$$

(c) $16\left(k_4 - \left(\frac{192}{913}\right)k_1\right) < k_3 - k_5$.

$$16\left(1800 - \frac{192}{913}(8380)\right) < 12140 - 80$$
$$16(1800 - 1762.278204) < 12060$$
$$603.5487 < 12060.$$

(d) $\frac{4}{55}\left(k_3 - \left(\frac{9}{256}\right)k_2\right) > k_4 - \left(\frac{192}{913}\right)k_1$.

$$\frac{4}{55}\left(12140 - \frac{9}{256}(230400)\right) > 1800 - \frac{192}{913}(8380)$$
$$\frac{4}{55}(4040) > 1800 - 1762.278204$$
$$293 > 37.7218$$

(e) $k_2 > 64\left(k_4 - \left(\frac{192}{913}\right)k_1\right)$

$$230400 > 64\left(1800 - \frac{192}{913}(8380)\right) = 2414.1950$$

(f) $k_2 > \left(\frac{2048}{913}\right) k_1$.

$$230400 > \frac{2048}{913}(8380) = 18797.6342$$

(g) $k_3 - k_5 > \left(\frac{512}{913}\right) k_1$

$$12140 - 80 > \frac{512}{913}(8380)$$
$$12060 > 4699.4085$$

(h) $k_3 > \frac{1}{8}\left(k_5 + 8k_1\right)$.

$$12140 > \frac{1}{8}\left(80 + 8(8380)\right) = 8390.$$

(i) $k_4 < (\frac{1}{8})(k_5 + 8k_1)$.

$$1800 < \frac{1}{8}(80 + 8(8380)) = 8390.$$

This shows that k_1 to k_5 satisfy $(a) - (i)$. Thus, by letting $b = 1$ we have the following:

(H1) $f(w) > \frac{34324480}{913}$ for $w \in [1800, 12140]$,

(H2) $f(w) < 460800$ for $w \in [0, 12140]$,

(H3) $f(w) < 96480$ for $w \in [0, 12140]$,

(H4) $f(w) > \frac{4408320}{913}$ for $w \in [0, 1800], and$

(H5) $f(w) < \frac{413696}{11}$ for $w \in [8390, 12140]$.

Then the constant function $f(w) = 37600$ satisfies the conditions mentioned above. Therefore, Theorem 3.1 guarantees the existence x^* of a positive symmetric solution of the boundary value problem

$$x'' + 37600 = 0,$$

$$x(0) = 0 = x(1),$$

with $x^* \in A(\beta, 12140, \alpha, 1800)$.

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